Dynamic Monopolies and Vaccination

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Joint with Bessy, Dourado, Ehard, Rautenbach
Dynamic Monopolies

Informal Definition

Dynamic monopolies are a simple graph-theoretical model for various types of viral processes in networks. Examples for things that can spread include opinions, computer viruses, diseases, products, habits, ...
Dynamic Monopolies

Informal Definition

Dynamic monopolies are a simple graph-theoretical model for various types of viral processes in networks. Examples for things that can spread include opinions, computer viruses, diseases, products, habits, etc.
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Dynamic Monopolies

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...examples for things that can spread...

- opinions,
- computer viruses,
- diseases,
- products,
- habits,
- ...
Dynamic Monopolies

(picture taken from www.quantamagazine.org)
Dynamic Monopolies

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Dynamic Monopolies

Let $G$ be a graph.

Let $\tau: V(G) \to \mathbb{Z}$ be a threshold function.

Let $D \subseteq V(G)$.

The hull $H(G, \tau)(D)$ of $D$ in $(G, \tau)$ is obtained as follows:

$H \leftarrow D$;

while $|N_G(u) \cap H| \geq \tau(u)$ for some $u \in V(G) \setminus H$ do

$H \leftarrow H \cup \{u\}$;

end

$H(G, \tau)(D) \leftarrow H$;

return $H(G, \tau)(D)$;
Dynamic Monopolies

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end

$H_{(G,\tau)}(D) \gets H;$

return $H_{(G,\tau)}(D);$
Dynamic Monopolies

Definition

\[ \text{dyn}(G, \tau) = \min \{ |D| : D \subseteq V(G) : H(G, \tau)(D) = V(G) \} \]

\[ \text{dyn}(G, 0) = 0 \]

\[ \text{dyn}(G, 1) = \text{number of components of } G \]

\[ \text{dyn}(G, d_G) = \text{minimum order of a vertex cover of } G \]

\[ \text{dyn}(G, d_G - 1) = \text{minimum order of a feedback vertex set of } G \]

\[ D \text{ is a dynamic monopoly of } (G, \tau) \leftrightarrow V(G) \setminus D \text{ is a } (d_G - \tau)\text{-degenerate set in } G. \]
Dynamic Monopolies

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### Dynamic Monopolies

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\[V(G) \setminus D \text{ is a } (d_G - \tau)\text{-degenerate set in } G.\]
Theorem (Chen ’09, P et al. ’11)

Determining $\text{dyn}(G, 2)$ is NP-hard.
Dynamic Monopolies

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Determining $\text{dyn}(G, 2)$ is NP-hard.

...even hard to approximate.
Dynamic Monopolies

A simple reduction algorithm for trees...
Dynamic Monopolies

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Dynamic Monopolies

A simple reduction algorithm for trees...

\[ \tau(v) \leq \tau(u) = s - v \tau(v) - 1 \]
Dynamic Monopolies

A simple reduction algorithm for trees...

\[
\begin{align*}
\tau(v) &\leq 0 = s_v \tau(v) - 1 \\
\tau(u) &
\end{align*}
\]
Dynamic Monopolies

A simple reduction algorithm for trees...

\[ \tau(v) \leq 0 \]
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\[ \tau(v) = \tau(u) \leq 0 \]
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\[ \tau(v) - 1 \]
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A simple reduction algorithm for trees...

\[ \tau(v) \geq 2 \]
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\[ \tau(v) \geq 2 \]

\[ \tau(u) \geq 2 = 1 \]
Dynamic Monopolies

A simple reduction algorithm for trees...

\[
\tau(v) = \tau(u) \geq 2 + 1
\]
A simple reduction algorithm for trees...

\[
\tau(v) = \begin{cases} 
\tau(u) \geq 2 & \text{if } \tau(v) \geq 2 \\
1 & \text{if } \tau(v) = 1 
\end{cases}
\]
Dynamic Monopolies

A simple reduction algorithm for trees...

\[ \tau(v) = 1 \]

\[ \tau(u) \]

\[ \tau(v) \]

\[ \text{Diagram: } v \quad \tau(v) \quad u \quad \tau(u) \]
Dynamic Monopolies

A simple reduction algorithm for trees...

\[
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Dynamic Monoplies

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Dynamic Monopolies

**Theorem (Chen ’09, P et al. ’11)**

For a given pair \((T, \tau)\), where \(T\) is a tree, \(\text{dyn}(T, \tau)\) can be determined in linear time.
Dynamic Monopolies

Two extensions of this result:
Dynamic Monopolies

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**Theorem (P et al. ’11)**

For a given pair \((G, \tau)\), where \(G\) has blocks of bounded order, \(\text{dyn}(G, \tau)\) can be determined in polynomial time.

**Theorem (Ben-Zwi et al. ’11)**

For a given pair \((G, \tau)\), where \(G\) has order \(n\) and treewidth \(w\), \(\text{dyn}(G, \tau)\) can be determined in \(nO(w)\) time. Furthermore, it is “highly unlikely” that \(\text{dyn}(G, \tau)\) can be determined in \(n^{o(\sqrt{w})}\) time.

The last result suggests that \(\text{dyn}(G, \tau)\) might only be tractable for tree-structured graphs.
Dynamic Monopolies

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Dynamic Monopolies

The key observation for another extension:

Lemma (P et al. '11)

If $(G, \tau)$ is such that $G$ is a 2-connected chordal graph and $\tau \leq 2$, then \{u, v\} is a dynamic monopoly for $(G, \tau)$ for every edge uv of G.
Dynamic Monopolies
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**Proof:**
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\begin{array}{c}
\end{array}
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**Proof:**

\[
\text{Diagram}
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**Proof:**

[Diagram showing a 2-connected chordal graph with a dynamic monopoly.]
Dynamic Monopolies

The key observation for another extension:

**Lemma (P et al. ’11)**

If $(G, \tau)$ is such that
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**Proof:**

![Diagram](image)
Dynamic Monopolies

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- \(G\) is a 2-connected chordal graph and
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then \(\{u, v\}\) is a dynamic monopoly for \((G, \tau)\) for every edge \(uv\) of \(G\).

**Proof:**

[Diagram of a 2-connected chordal graph with a dynamic monopoly indicated.]
For a given pair \((G, \tau)\), where

- \(G\) is chordal and
- \(\tau \leq 2\),

\(\text{dyn}(G, \tau)\) can be determined in polynomial time.
Dynamic Monopolies

Lemma (Chiang et al. '13)

Let $t$ be a non-negative integer. If $(G, \tau)$ is such that

- $G$ is a $t$-connected chordal graph and
- $\tau \leq t$,

then $C$ is a dynamic monopoly for $(G, \tau)$ for every clique $C$ of order $t$. 

In particular, $\text{dyn}(G, \tau) \leq t$. 

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Let $t$ be a non-negative integer. If $(G, \tau)$ is such that
- $G$ is a $t$-connected chordal graph and
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$$\text{dyn}(G, \tau) \leq t.$$
Dynamic Monopolies

Problem

Is there a polynomial time algorithm that determines

\[ \text{dyn}(G, \tau) \]

for a given pair \((G, \tau)\) such that

- \(G\) is chordal, and
- \(\tau\) is bounded?
Let $t$ be a non-negative integer. For a given pair $(G, \tau)$, where

- $G$ is an interval graph, and
- $\tau \leq t$,

$\text{dyn}(G, \tau)$ can be determined in polynomial time.
Dynamic Monopolies

Theorem (BEPR '18)

For a given triple \((G, \tau, k)\), where

- \(G\) is a chordal graph,
- \(\tau\) is a threshold function for \(G\), and
- \(k\) is a positive integer,

it is NP-complete to decide whether \(\text{dyn}(G, \tau) \leq k\).
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Dynamic Monopolies
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Let $G$ be an interval graph of order $n$, and let $\tau \leq t$ be a threshold function.
Dynamic Monopolies

Let $G$ be an interval graph of order $n$, and let $\tau \leq t$ be a threshold function. Let $(I(u))_{u \in V(G)}$ be an interval representation using closed intervals with distinct endpoints $x_1 < x_2 < \ldots < x_{2n}$. 

For $c_i = |C_i|$, we have $|c_i - c_{i+1}| = 1$. 

Let $j_1 < j_2 < \ldots < j_{k-1}$ be the indices $i$ with $c_i < \min\{c_i - 1, c_i + 1, t\}$ and let $j_k = 2n-1$. 

Dynamic Monopolies

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\[ x_i \quad x_{i+1} \]
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Every minimal vertex cut of $G$ is a clique of the form

$$C_i = \left\{ u \in V(G) : [x_i, x_{i+1}] \subseteq I(u) \right\}.$$
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For $c_i = |C_i|$, we have $|c_i - c_{i+1}| = 1$. 
Dynamic Monopolies

Let $G$ be an interval graph of order $n$, and let $\tau \leq t$ be a threshold function. Let $(l(u))_{u \in V(G)}$ be an interval representation using closed intervals with distinct endpoints $x_1 < x_2 < \ldots < x_{2n}$.

Every minimal vertex cut of $G$ is a clique of the form

$$C_i = \left\{ u \in V(G) : [x_i, x_{i+1}] \subseteq l(u) \right\}.$$

For $c_i = |C_i|$, we have $|c_i - c_{i+1}| = 1$.

Let $j_1 < j_2 < \ldots < j_{k-1}$ be the indices $i$ with

$$c_i < \min \left\{ c_{i-1}, c_{i+1}, t \right\}$$

and let $j_k = 2n - 1$. 
Dynamic Monopolies

\[ V_i = C_1 \cup \cdots \cup C_j, \quad G_i = G[V_i], \quad B_i = C_j. \]

\[ |B_i| < t. \]

No vertex in \( V_i \setminus B_i \) has a neighbor in \( V(G) \setminus V_i \).

\[ \partial V_i = (V_i \setminus V_{i-1}) \cup B_{i-1}, \quad \partial G_i = G[\partial V_i]. \]
Dynamic Monopolies

Let $V_i = C_1 \cup \cdots \cup C_{j_i}$, $G_i = G[V_i]$, and $B_i = C_{j_i}$.
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| \( B_i \) | < \( t \).
Dynamic Monopolies

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Dynamic Monopolies

Let $V_i = C_1 \cup \cdots \cup C_{j_i}$, $G_i = G[V_i]$, and $B_i = C_{j_i}$.

$|B_i| < t$. No vertex in $V_i \setminus B_i$ has a neighbor in $V(G) \setminus V_i$.

Let $\partial V_i = (V_i \setminus V_{i-1}) \cup B_{i-1}$, and $\partial G_i = G[\partial V_i]$. 
Dynamic Monopolies

Claim

Each $\partial G_i$ is either a clique of order at most $t$ or $t$-connected.
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Proof:
Dynamic Monopolies

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Each $\partial G_i$ is either a clique of order at most $t$ or $t$-connected.

Proof: Suppose $t = 3$. 
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\[
\begin{array}{c}
\downarrow \\
| \\
\downarrow \\
| \\
\end{array}
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Dynamic Monopolies

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Each \( \partial G_i \) is either a clique of order at most \( t \) or \( t \)-connected.

Proof: Suppose \( t = 3 \).
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![Diagram of a grid with arrows indicating connections between nodes]
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\[
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\[ \square \]
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(jump a little!?)
Dynamic Monopolies

A local cascade for $G_i$ is a triple $(X_i, \prec_i, \rho_i)$, where

(i) $X_i$ is a subset of $B_i$,
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(iii) $\rho_i : B_i \setminus X_i \to \{0, 1, \ldots, n\}$.

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For a local cascade \((X_i, \prec_i, \rho_i)\) for \(G_i\), let

\[ \text{dyn}_i(X_i, \prec_i, \rho_i) \]

be the minimum order of a subset \(Y_i\) of \(V_i \setminus B_i\) such that the following conditions hold:
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Dynamic Monopolies

Claim

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Vaccination

\[ \text{dyn}(G, \tau) \] may be considered a measure of network vulnerability. Given a budget \( b \in \mathbb{Z} \geq 0 \), we want to minimize this vulnerability by maximizing \( \text{dyn}(G, \tau) \).

**Scenario 1**
By increasing the threshold value of \( b \) vertices beyond their degree. (Note that the vaccinated vertices belong to every dynamic monopoly and still participate in the spreading.)

**Scenario 2**
By removing \( b \) vertices. (The vaccinated vertices no longer participate in the spreading.)

**Scenario 3**
By increasing the threshold values of individual vertices subject to vertex-dependent lower and upper bounds, and fixing the total increase to \( b \). (Partial/imperfect immunization.)
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<thead>
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</tr>
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Theorem (Khoshkhah et al. ’15)

Requiring $0 \leq \tau \leq d_G$ in the above setting, the problem becomes NP-hard for planar graphs but can be solved efficiently for trees.
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\[
vacc_3(G, \tau, \iota_{\text{max}}, b) = \max \left\{ \text{dyn}(G, \tau + \iota) : \iota \in \mathbb{Z}^{V(G)}, \right. \\
\quad \left. 0 \leq \iota \leq \iota_{\text{max}}, \text{ and } \iota(V(G)) = b \right\}
\]
Vaccination

Theorem (BDEPR '18)

Given a tree $T$ of order $n$, $\tau$, $b$, and $\iota_{\text{max}}$, $\text{vacc}_1(T, \tau, b)$ and $\text{vacc}_3(T, \tau, b)$ can be determined in $O(n^2(b + 1)^2)$ time, and $\text{vacc}_2(T, \tau, b)$ can be determined in $O(n^3(b + 1)^2)$ time.
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Thank you for the attention!