A proof of the Erdős Sands Sauer Woodrow conjecture

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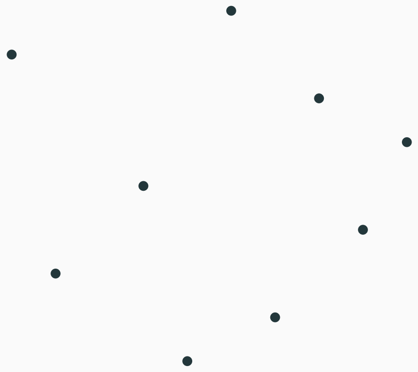
Covering points in $\mathbb{R}^2$



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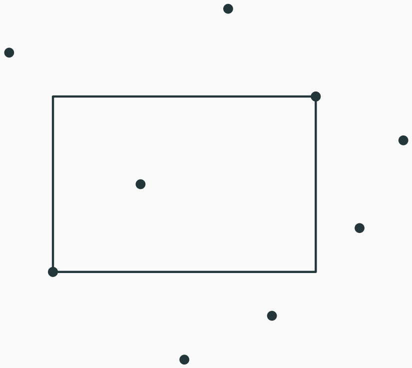

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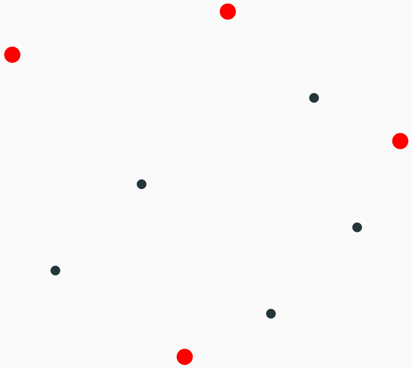

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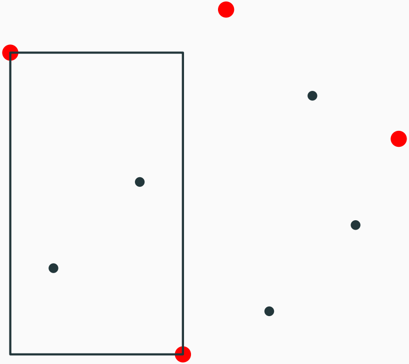

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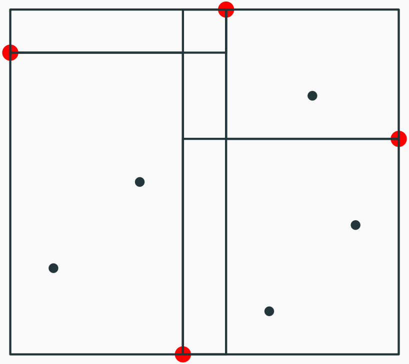
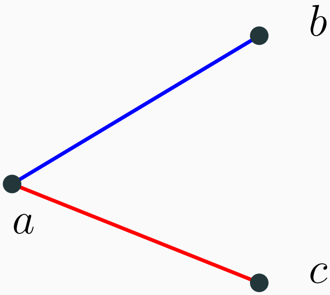


Bárány and Lehel

Theorem (Bárány and Lehel, 87)
Every finite subset X of $\mathbb{R}^d$ can be covered by $h(d)$ X-boxes

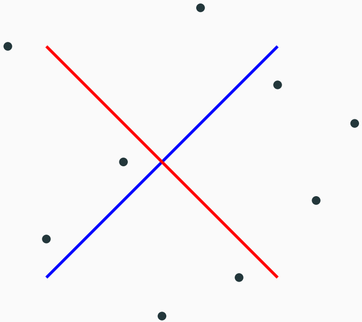
- Equivalent to taking all the boxes between $g(d)$ points
- Easy in $\mathbb{R}^2$
- Already hard in $\mathbb{R}^3$ (best known upper bound is 64 boxes)
- Lower bound of $h(d) \geq 2^{2^{d-1}}$

Two types of edges



- ab is blue because $y_a < y_b$
- ac is red because $y_a > y_c$

Bicoloured clique



- 2 directions

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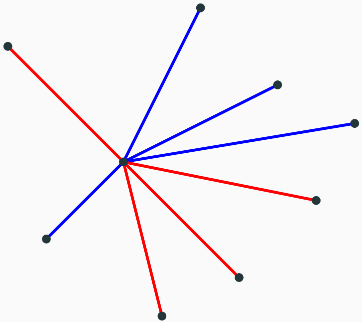
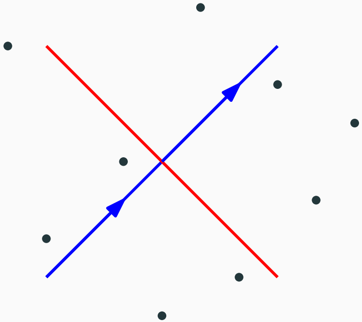


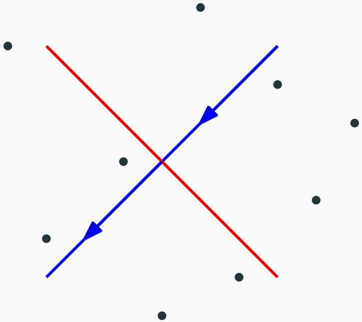
- 2 directions

Bicoloured tournaments



- Each direction has two possible orientations

Bicoloured tournaments



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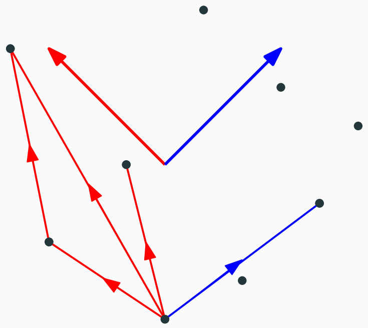
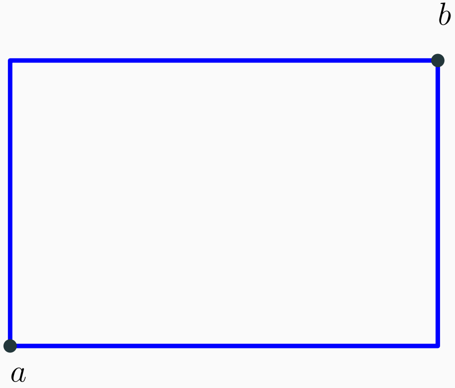
Bicoloured tournaments



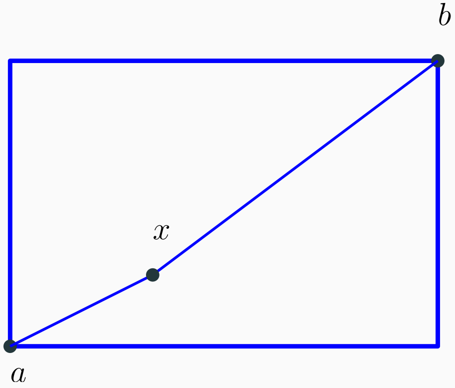
- Any orientation of the directions gives a tournament

Blue boxes



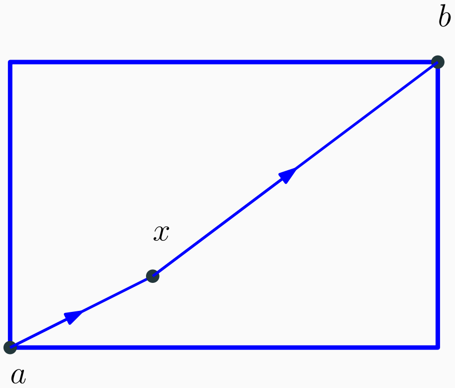
- There is two type of boxes: blue and red

Blue boxes



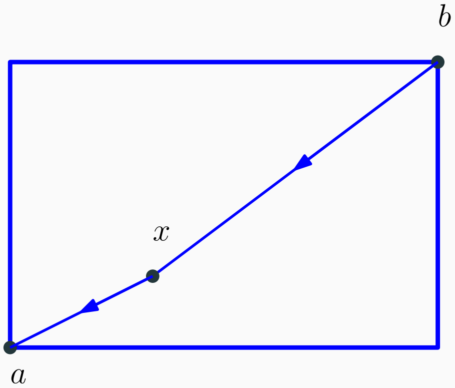
- A vertex x is in the box if ax and bx are blue

Blue boxes



- Any natural orientation of the blue direction will put x in the middle of a blue ab directed path

Blue boxes



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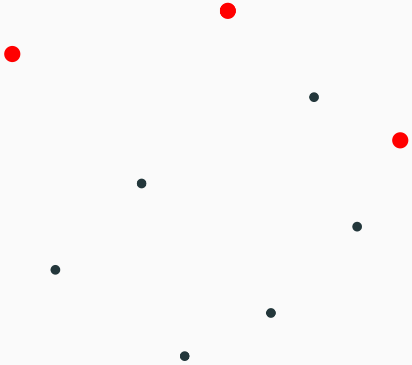
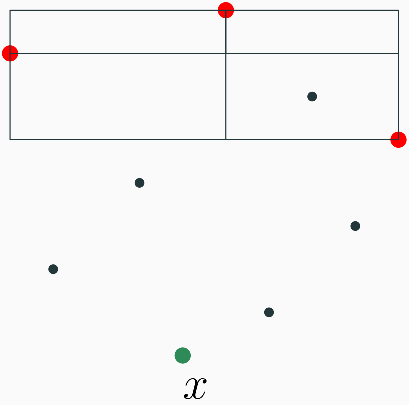
Dominating points



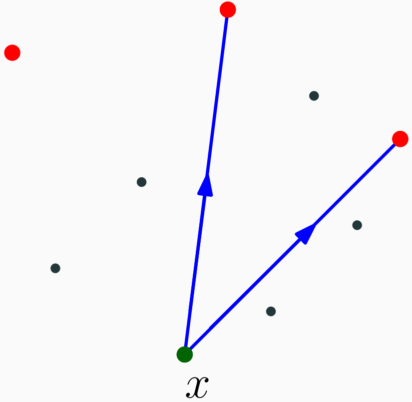
- Suppose the red vertices don't cover every vertices.

Dominating points



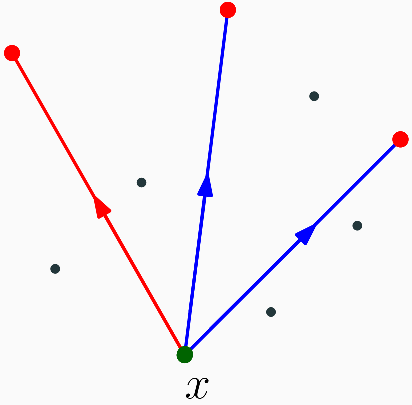
- Let x be an uncovered vertex.

Dominating points



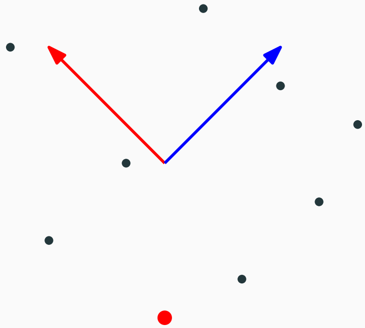
- In any orientation of blue x has out or in-degree 0 with the red vertices.

Dominating points



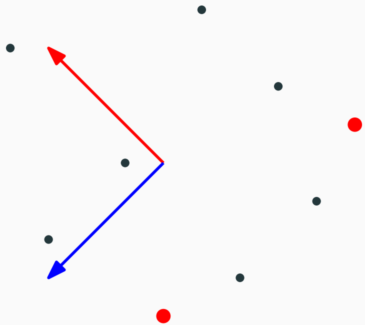
- In one of the tournaments, x dominates the red vertices.

Strategy



- Pick a dominating set in every tournament

Strategy



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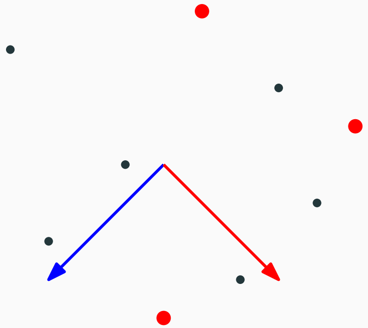

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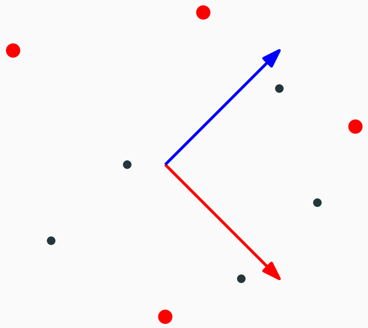
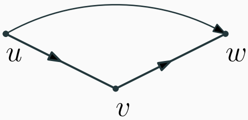


- It gives the four points

Definitions

- A digraph D is *transitive* if $uv, vw \in A(D) \Rightarrow uw \in A(D)$.



- A digraph D is *k-transitive* if there exists a partition of $A(D)$ into k transitive digraphs
- $S \subset V(D)$ is *dominating* if for all $x \in V(D) \setminus S$, $\exists s \in S$ such that $sx \in A(D)$
- $\gamma(D)$ is the size of the smallest dominating set

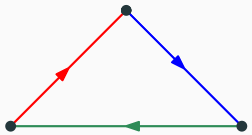
- In dimension D : 2^{d-1} directions
- This gives $2^{2^{d-1}}$ tournaments, all 2^{d-1}-transitive

We used the following fact:

Lemma

Every 2-transitve tournament T is such that $\gamma(T) = 1$

Already false for 3-transitive tournaments.



Conjecture

Conjecture (Gyarfas)
For every k-transitive tournament T, $\gamma(T) \leq f(k)$.

- Weakening of a conjecture of Erdős Sands Sauer and Woodrow.
- The case $k = 2$ was proved in the paper of Sands, Sauer and Woodrow in 82
- The case $k = 3$ stayed open until now

Domination in Tournaments

- A random tournament T_n has $\gamma(T_n) = O(log(n))$.

Proposition

Let T be a tournament on n vertices, then :

- *There exists a set S of $log(n)$ vertices dominating T*
- *There exists a set S of $f(\epsilon)$ vertices dominating $n(1-\epsilon)$ vertices of T*

Proof

Proposition (Fisher, Ryan 1995)
For any tournament T, there exists a probability distribution w on $V(T)$ such that, for every x, $w(N^-[x]) \geq 1/2$.

Pick a set S of k vertices at random according to w:

- $P(x \leftarrow S) \geq 1 - (1/2)^k$
- $E(N^+[S]) = \sum_x P(x \leftarrow S) \geq n(1 - (1/2)^k)$

And thus there exists such a set S.

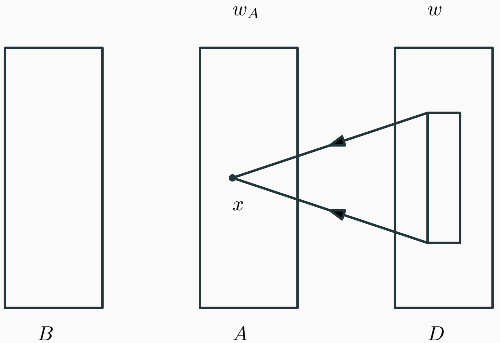
Dominating by transitivity

Lemma

D be a transitive digraph. Suppose there exists $B, A \subset D$ with:

- $w : D \to [0,1]$ *s.t* $w(N^-[x]) > \epsilon$, $\forall x \in A$
- $w_A : A \to [0,1]$ *s.t* $w_A(N^-[x]) > \epsilon$, $\forall x \in B$

Then B is dominated by a set S of $g(\epsilon)$ vertices of D.



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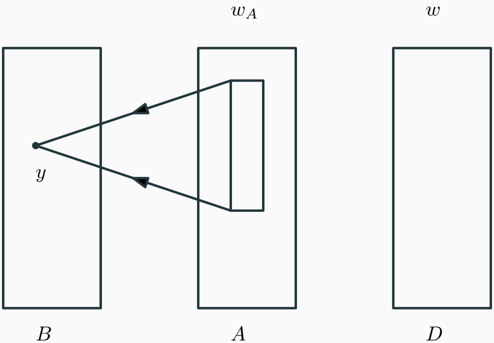
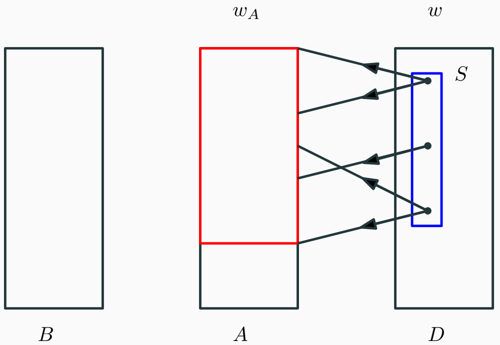
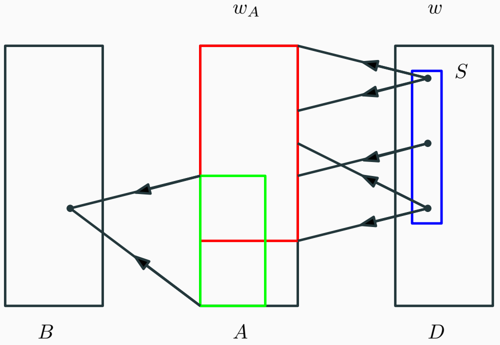


Proof of the lemma



- Pick $S \subset D$ s.t $w_A(N^+(S)) > (1 - \epsilon)$
- For every $x \in B$, $N^-[x] \cap N^+(S) \neq \emptyset$
- By transitivity, S dominates B.

Proof of the lemma



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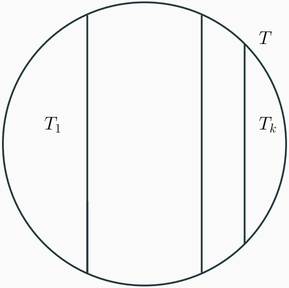
Winning strategy on tournaments

Proposition (Fisher, Ryan 1995)

For any tournament T, there exists a probability distribution w on $V(T)$ such that, for every x, $w(N^-[x]) \geq 1/2$.

Lemma

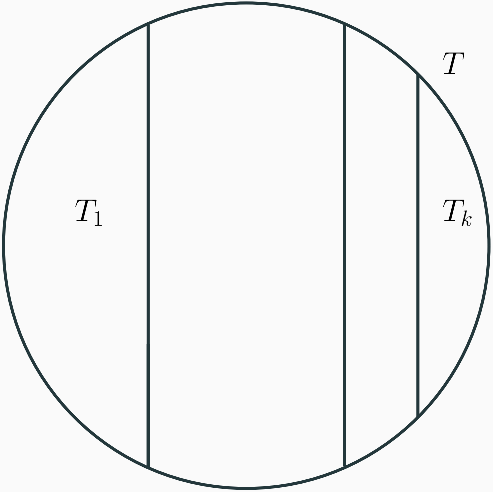
Let T be a k-transitive tournament, there exists a probability distribution w on $V(T)$ and a partition $T_1, T_2, \ldots, T_k$ of $V(T)$ such that for every i and $x \in T_i$, $w(N_i^-(x)) \geq 1/2k$.



Proof 1/3

Theorem (Bousquet, L., Thomassé)

For every k-transitive tournaments T, $\gamma(T) \leq k^{k+1} * g(1/2k)$.



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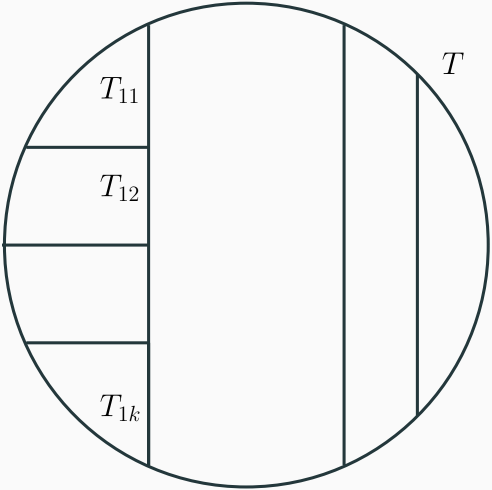
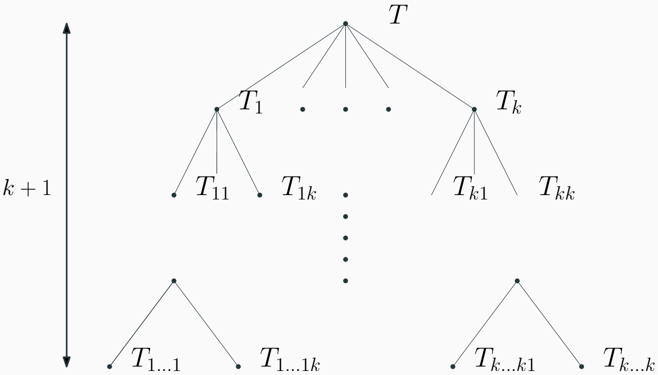


Proof 2/3

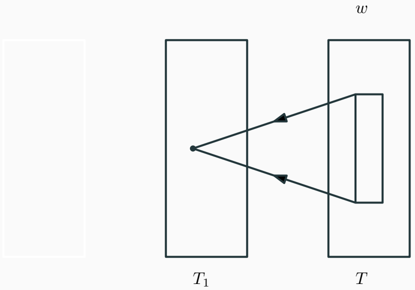


- Each level is a partition of $V(T)$
- For each $T_{i_1\ldots i_{k+1}}$ one colour appears twice

Proof 3/3

$T_{1,1}$, T_1, w and w_1 are such that, in the digraph induced by colour 1:

- $w : T \to [0,1]$ s.t $w(N^-[x]) > 1/2k$, $\forall x \in T_1$
- $w_1 : T_1 \to [0,1]$ s.t $w_1(N^-[x]) > 1/2k$, $\forall x \in T_{1,1}$



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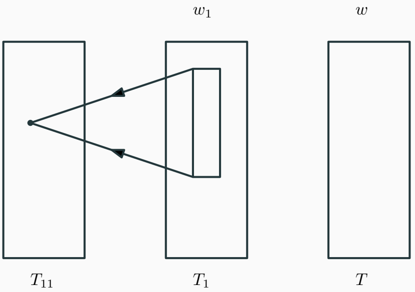

And thus T_{11} is dominated by $g(1/2k)$ vertices of T

Erdos-Sands-Sauer-Woodrow

Problem

For each k, is there a (least) positive integer $f(k)$ so that every finite tournament whose edges are coloured with n colours contains a set S of $f(k)$ vertices with the property that for every vertex u not in S there is a monochromatic path from u to a vertex of S?

By replacing each colour class by it's transitive closure we obtain the following conjecture:

Conjecture

For every k, there exists an integer $f(k)$ such that if T is a complete multidigraph whose arcs are the union of k quasi-orders, then $\gamma(T) \leq f(k)$.

Conjecture

For every k, there exists an integer $f(k)$ such that if D is a multidigraph whose arcs are the union of k quasi-orders, then D has a dominating set which is the union of $f(k)$ stables sets.

- Only case known is $k = 2$, proved by Sands, Sauer and Woodrow in 82
- Our proof works for the class of bounded α

Thank you!