Zero Forcing

Dieter Rautenbach

Universität Ulm

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Joint work with M. Gentner, L.D. Penso, and U.S. Souza

Graphs and Matrices

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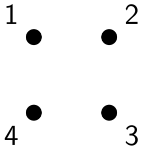

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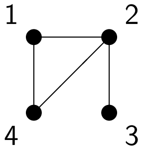

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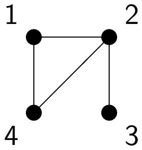
$G(A) =$ 

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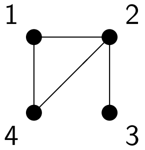
$G(A) =$ 

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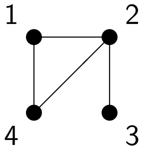
$G(A) =$ 

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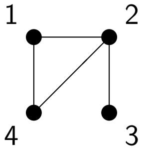

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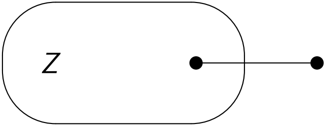

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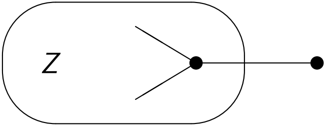

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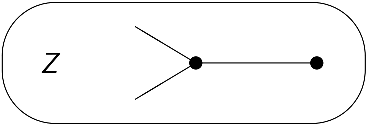

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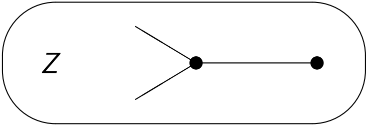

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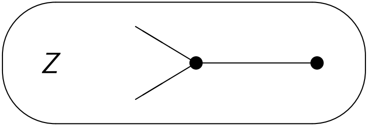

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$$Z(G) = \min\{|Z| : \mathcal{F}(Z) = V(G)\}$$

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AIM Minimum Rank - Special Graphs Work Group

Barioli, Barrett, Butler, Cioaba, Cvetkovic, Fallat, Godsil, Haemers, Hogben, Mikkelson, Narayan, Pryporova, Sciriha, So, Stevanovic, van der Holst, Meulen, Wehe

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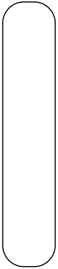

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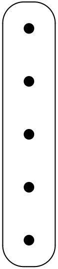

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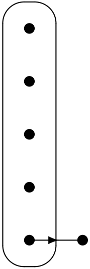

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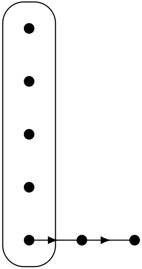

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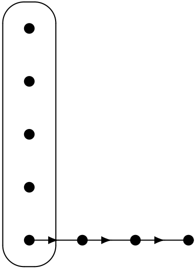

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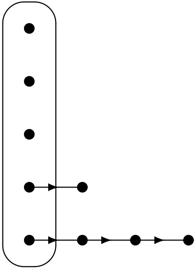

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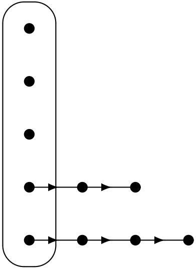

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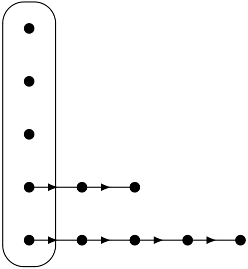

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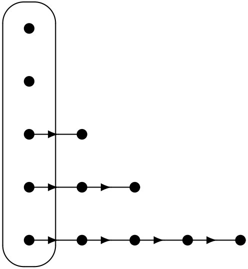

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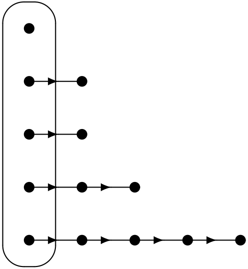

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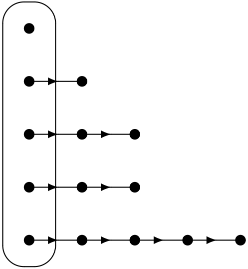

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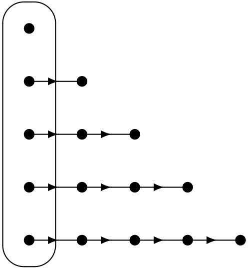

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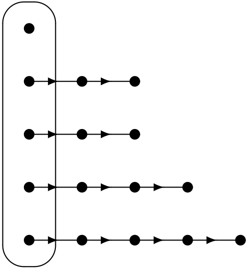

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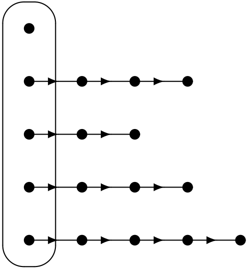

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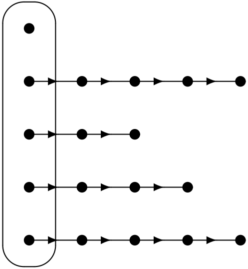

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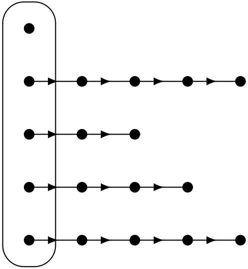

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For a graph G, let $P(G)$ be the minimum number of disjoint induced paths $P_1, \ldots, P_k$ in G with $V(G) = V(P_1) \cup \ldots \cup V(P_k)$.

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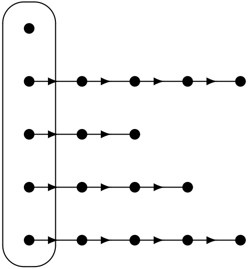

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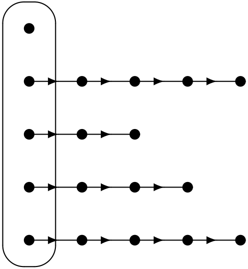

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- Both parameters are computationally hard
  (Aazami '08, Fallat et al. '16, Le, Le, and Müller '03).

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Theorem (Amos, Caro, Davila, and Pepper '15)

Let G be a graph of order n, maximum degree Δ, and minimum degree at least 1.

(i) $Z(G) \leq \frac{\Delta n}{\Delta+1}$.

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This conjecture is true.

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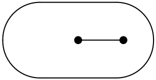

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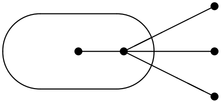

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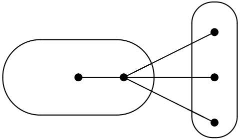

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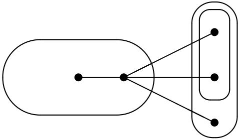

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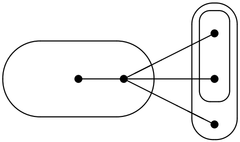

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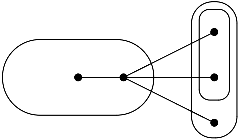

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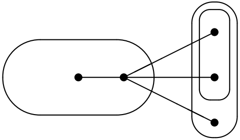

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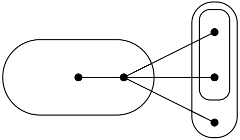

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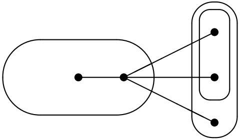

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- (ii) of the Theorem of Amos et al. implies (i), because

$$\frac{(\Delta - 2)n + 2}{\Delta - 1} \leq \frac{\Delta n}{\Delta + 1}$$

for $n \geq \Delta + 1$.

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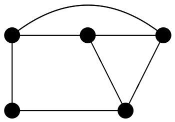

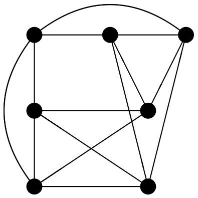

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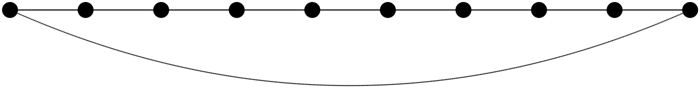

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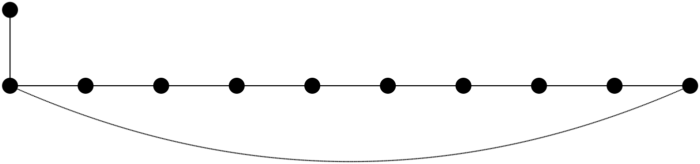

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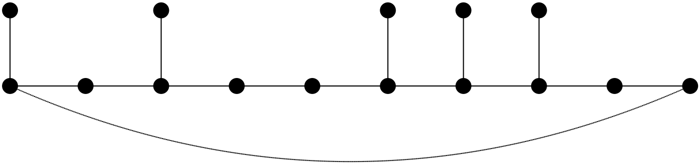

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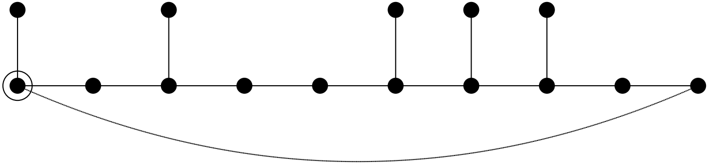

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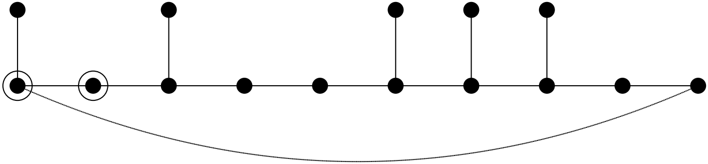

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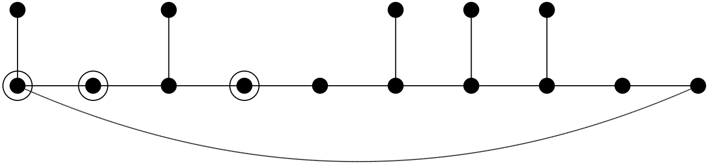

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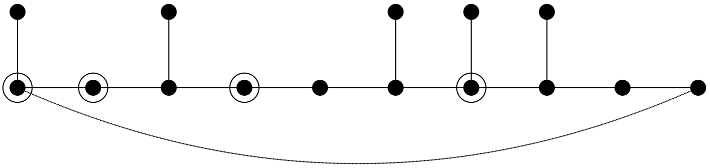

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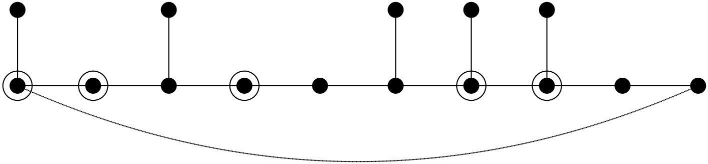

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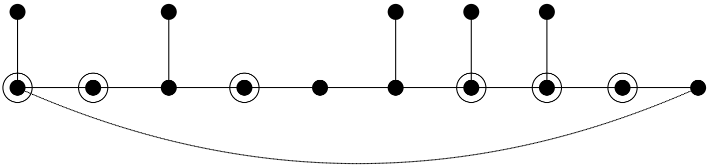

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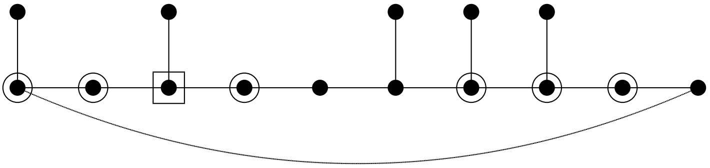

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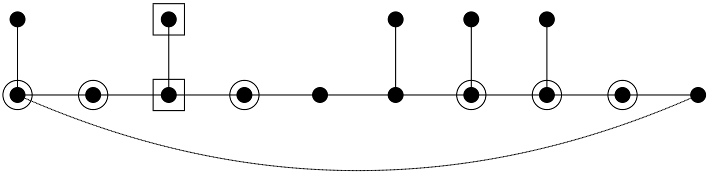

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Proof (for $\Delta = 3$ and $g \geq 5$): We need to find a set Z_0 with

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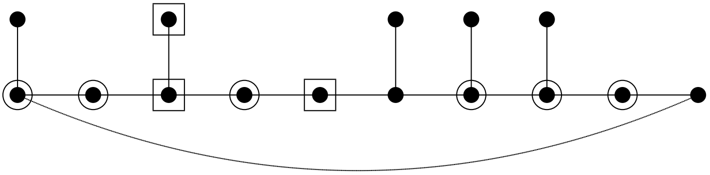

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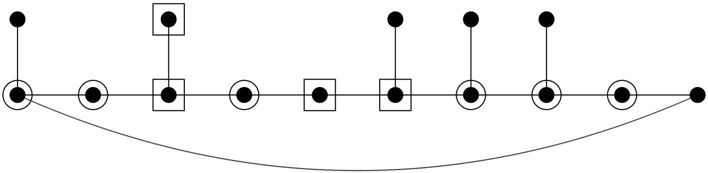

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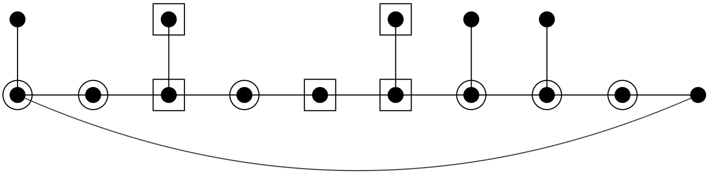

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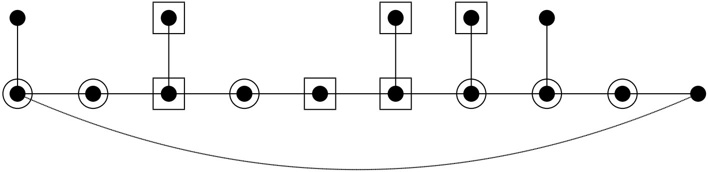

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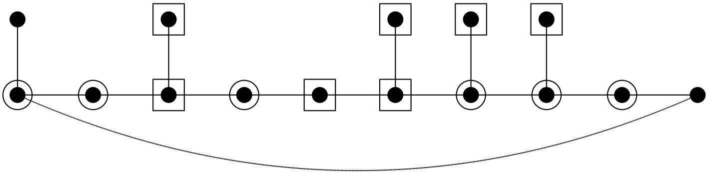

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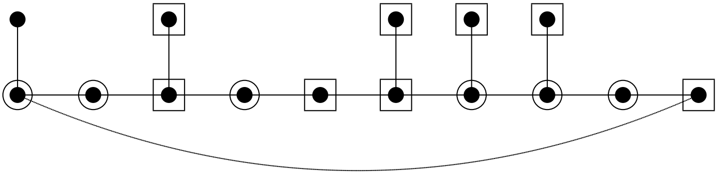

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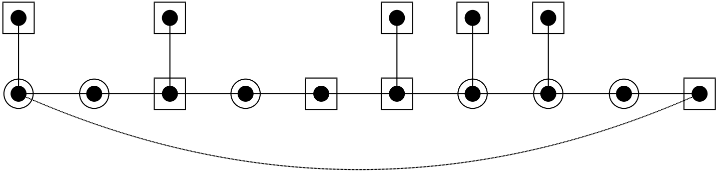

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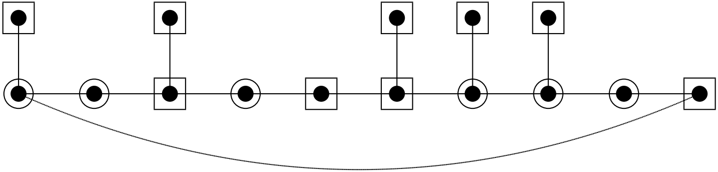

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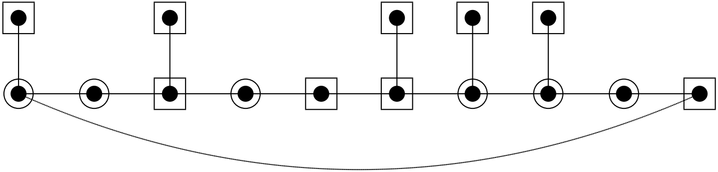

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Conjecture (GR '16)

If G is a connected graph of order n and maximum degree 3, then

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Theorem (GR '16)

If G is a connected graph of order n, maximum degree 3, and girth at least 5, then

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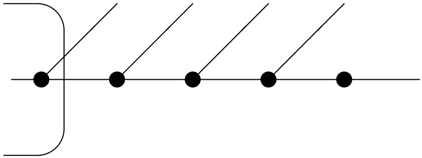

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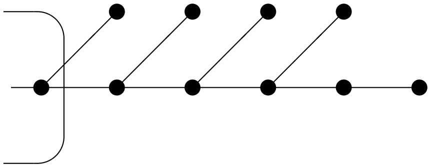

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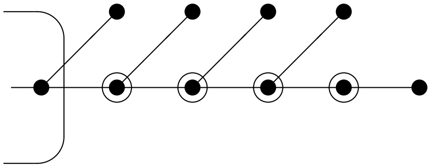

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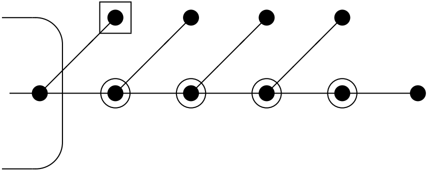

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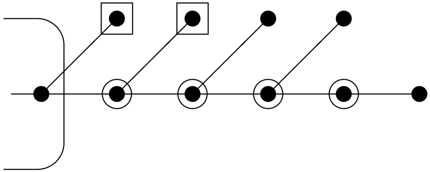

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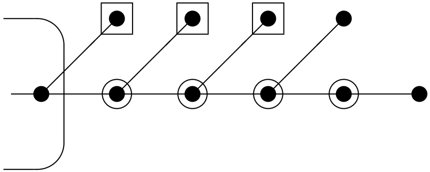

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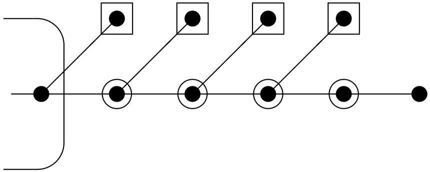

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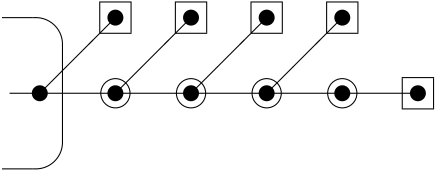

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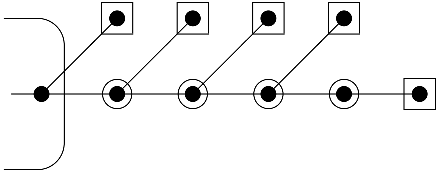

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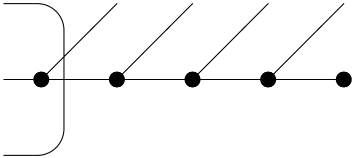

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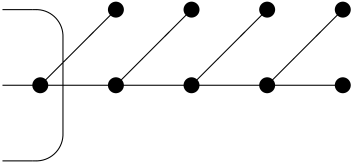

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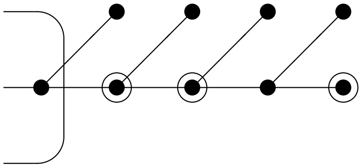

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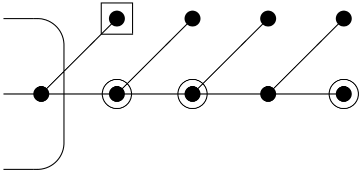

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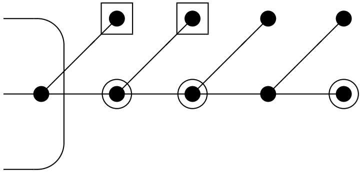

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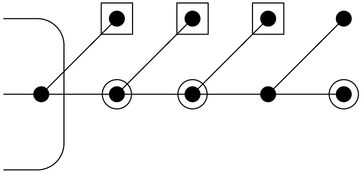

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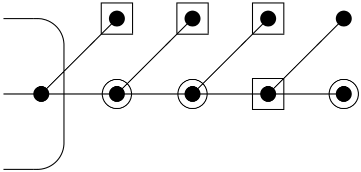

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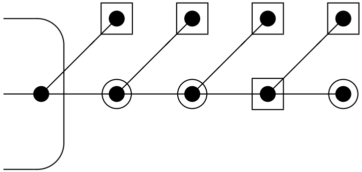

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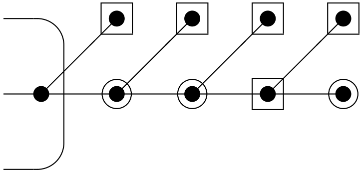

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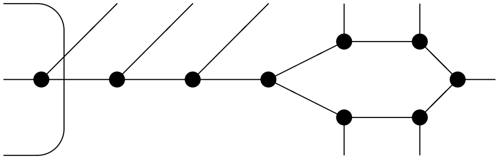

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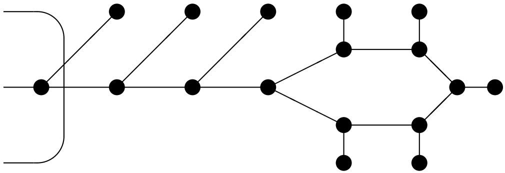

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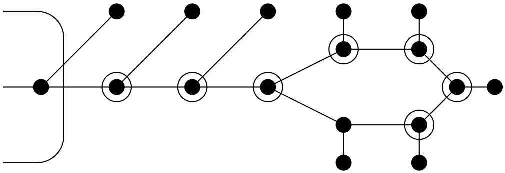

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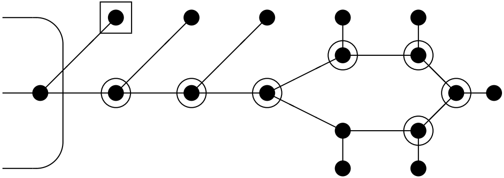

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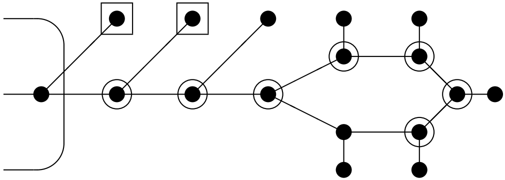

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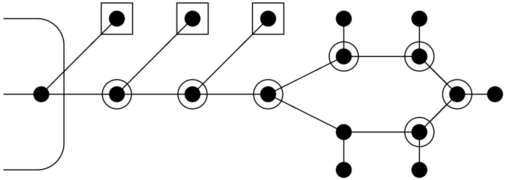

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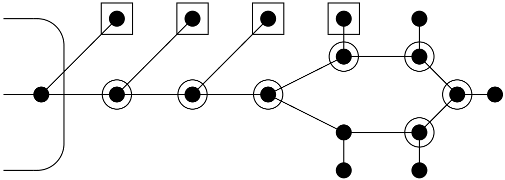

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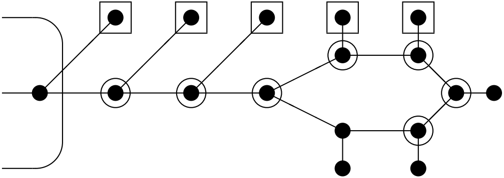

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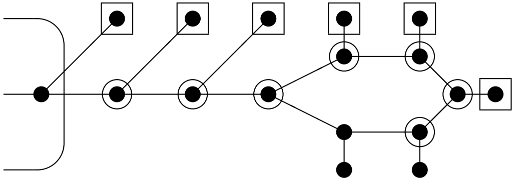

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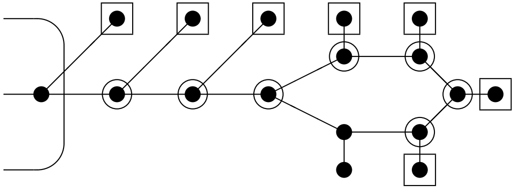

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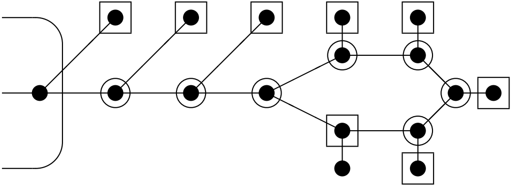

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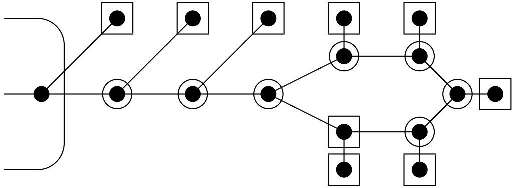

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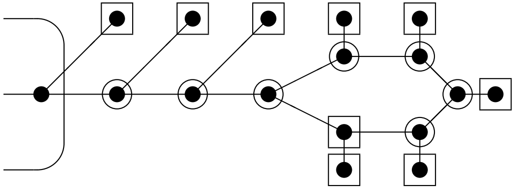

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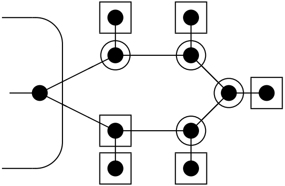

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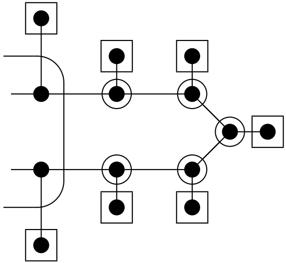

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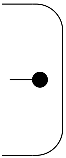

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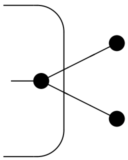

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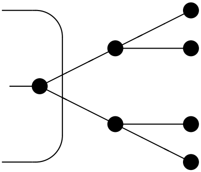

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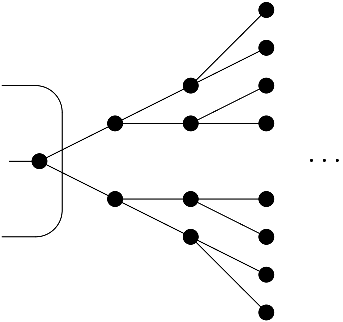

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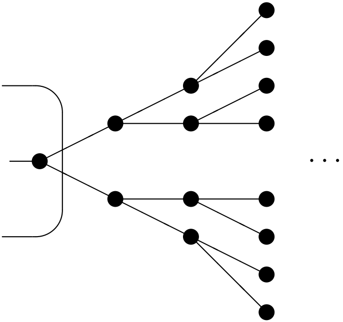

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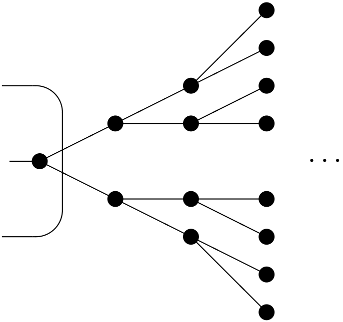

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Theorem (GR '16)

If G is a graph, then

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Corollary (GR '16)

If G is a r-regular graph of order n and girth at least 5, then

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$$Z(G) \leq \left(\prod_{i=1}^{r} \left(1 - \frac{1}{ri+1} \right) \right) n = \left(1 - \frac{H_r}{r} \right) n + O\left(\left(\frac{H_r}{r} \right)^2 \right) n.$$

$$H_r = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r} \sim \ln r$$

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(i) *If $\delta \geq 3$ and $g \geq 4$, then $Z(G) \geq \delta + 1$.*

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Conjecture (Kenter and Davila '14)

If $\delta \geq 2$ and $g \geq 3$, then

$$Z(G) \geq (g-2)(\delta-2)+2.$$

Lower Bounds

For $g \geq 7$ and $\delta \geq \delta_g$, the conjecture follows using

- $Z(G) \geq tw(G)$ and
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Theorem (GR '16)

The conjecture holds for $g \in \{4, 5, 6\}$.

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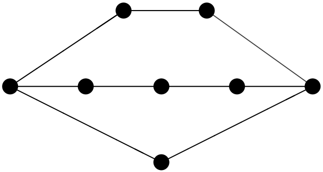

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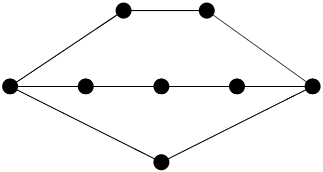

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Folklore

A graph is a cactus if and only if it is $\mathcal{F}$-free for

$$\mathcal{F} = \{K_4\} \cup \{\Theta(\ell_1, \ell_2, \ell_3) : \ell_1, \ell_2, \ell_3 \in \mathbb{N} \text{ and } \ell_2, \ell_3 \geq 2\}.$$

Hereditary Equality with $P(G)$

Theorem (GPRS '16)

If G is a graph such that every cycle of G is induced, then the following statements are equivalent.

(i) $G \in \mathcal{ZP}$.

(ii) G *is a cactus.*

(iii) G *is* $\mathcal{F}$*-free.*

The end

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Thank you for your attention!