Zero Forcing

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Joint work with M. Gentner, L.D. Penso, and U.S. Souza
Graphs and Matrices

Let $S(n) = \{ A \in \mathbb{R}^{n \times n} : A^T = A \}$. 

\[
\begin{bmatrix}
1 & 2 & 0 & 2 \\
2 & 6 & 3 & 2 \\
0 & 3 & 0 & 0 \\
2 & 2 & 0 & 1 \\
\end{bmatrix}
\]

$G(A) = \{ 1 2 \ 3 4 \}$. 

Let $S(G) = \{ A \in S(n) : G(A) = G \}$ and $M(G) = \max \{ n - \text{rg}(A) : A \in S(G) \} \leq \cdots$ (zero forcing)
Graphs and Matrices

Let $S(n) = \{A \in \mathbb{R}^{n \times n} : A^T = A\}$.
For $A \in S(n)$, let $G(A)$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set

$$\{ij : 1 \leq i < j \leq n \text{ and } a_{i,j} \neq 0\}.$$
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\end{pmatrix}
\]

\( G(A) \) =

\[
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
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\end{array}
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Let \( S(G) = \{ A \in S(n) : G(A) = G \} \) and \( M(G) = \max \{ n - \text{rg}(A) : A \in S(G) \} \leq \text{???(; zero forcing)} \).
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G(A) = \begin{pmatrix}
\bullet & \bullet & \bullet & \bullet \\
1 & 2 & 4 & 3
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$$\leq \quad ???$$
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$$M(G) = \max \{ n - \text{rg}(A) : A \in S(G) \} \leq ??? \quad (\sim \text{zero forcing})$$
Zero Forcing

For a graph $G$ and a set $Z \subseteq V(G)$, let $F(Z)$ be defined by the following procedure.

while $|N_G(u) \setminus Z| = 1$ for some $u \in Z$
do $Z \leftarrow Z \cup (N_G(u) \setminus Z)$
end

$F(Z) \leftarrow Z$

Definition (AIM group '08)
If $F(Z) = V(G)$, then $Z$ is a zero forcing set of $G$.

$Z(G) = \min \{ |Z| : F(Z) = V(G) \}$ is the zero forcing number of $G$.
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![Diagram of Zero Forcing](Z)
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Theorem (AIM group ‘08)

Let \( Z \) be a zero forcing set of a graph \( G \).

Let \( A \in \mathcal{S}(G) \) and let \( x \in \ker(A) \).

(i) If \( x_u = 0 \) for \( u \in Z \), then \( x = 0 \).

(ii) \( M(G) \leq Z(G) \).

Proof:

(i) If \( u \in Z \) is such that \( N_G(u) \setminus Z = \{ v \} \), then \( 0 = (Ax)_u = \sum_{w \in V(G)} a_{u,w}x_w = a_{u,v}x_v \), and hence \( x_v = 0 \).

(ii) Suppose \( n - \text{rg}(A) > |Z| \).

For \( U = \{ x \in \mathbb{R}^n : x_u = 0 \text{ for } u \in Z \} \),

\[
\dim(\ker(A) \cap U) = \dim(\ker(A)) + \dim(U) - \dim(\ker(A) + U) > |Z| + (n - |Z|) - n = 0,
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contradicting (i).
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(ii) Suppose $n - \rg(A) > |Z|$. 

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$$\dim(\ker(A) \cap U)$$
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$$\dim(\ker(A) \cap U) = \dim(\ker(A)) + \dim(U) - \dim(\ker(A) + U) > |Z| + (n - |Z|) - n.$$
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contradicting (i).
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Zero Forcing

For a graph $G$, let $P(G)$ be the minimum number of disjoint induced paths $P_1, \ldots, P_k$ in $G$ with $V(G) = V(P_1) \cup \ldots \cup V(P_k)$.

$Z(G) \geq P(G)$ with equality for forests (AIM group) and cacti (Row '11).

Both parameters are computationally hard (Aazami '08, Fallat et al. '16, Le, Le, and Müller '03).
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$Z(G) \geq P(G)$ with equality for forests (AIM group) and cacti (Row '11). Both parameters are computationally hard (Aazami '08, Fallat et al. '16, Le, Le, and Müller '03).
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Upper Bounds

Theorem (Amos, Caro, Davila, and Pepper '15)

Let $G$ be a graph of order $n$, maximum degree $\Delta$, and minimum degree at least 1.

(i) $Z(G) \leq \Delta n + 1$.

(ii) If $G$ is connected and $\Delta \geq 2$, then $Z(G) \leq (\Delta - 2)n + 2\Delta - 1$.

Conjecture (Amos, Caro, Davila, and Pepper '15)

The only extremal graphs for (ii) are $C_n$, $K_n$, and $K_{\Delta, \Delta}$.

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Lemma (GR '16)

Let $G$ be a connected graph of order $n$ and maximum degree $\Delta$ at least 3.

If there is some set $Z_0$ of vertices of $G$ such that $|Z_0| \leq \Delta - 2\Delta - 1 |F(Z_0)| + \alpha$, and $F(Z_0)$ induces a subgraph of $G$ without isolated vertices, then $Z(G) \leq \Delta - 2\Delta - 1 n + \alpha$. 
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Proof:

For some $i \geq 0$, let $Z_i$ be such that $|Z_i| \leq \Delta - 2\Delta - 1 |F(Z_i)| + \alpha$. 

Let $u \in F(Z_i)$ be such that $\emptyset \neq NG(u) \subseteq F(Z_i) = \{v\} \cup NG$.

If $Z_{i+1} = Z_i \cup NG$, then $|Z_{i+1}| = |Z_i| + |NG|$, $|F(Z_{i+1})| \geq |F(Z_i)| + |NG| + 1$, which implies $|Z_{i+1}| \leq \Delta - 2\Delta - 1 |F(Z_{i+1})| + \alpha$. 

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$\mathcal{F}(Z_i) \neq V(G)$. 
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$\mathcal{F}(Z_i) \neq V(G)$. Let $u \in \mathcal{F}(Z_i)$ be such that

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To obtain (ii) of the Theorem of Amos et al. choose

\[ Z_0 = N_G[u] \setminus \{v\} \]

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(ii) of the Theorem of Amos et al. implies (i), because 

$$\frac{(\Delta - 2)n + 2}{\Delta - 1} \leq \frac{\Delta n}{\Delta + 1}$$

for $n \geq \Delta + 1$. 
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Theorem (GR ‘16)

If $G$ is a connected graph of order $n$ and maximum degree $\Delta$ at least 3, then

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If $G$ is a connected graph of order $n$ and maximum degree $\Delta$ at least 3, then

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If $G$ is a connected graph of order $n$ and maximum degree $\Delta$ at least 3, then

\[ Z(G) \leq \frac{\Delta - 2}{\Delta - 1} n \]

if and only if $G \notin \{K_{\Delta+1}, K_{\Delta,\Delta}, K_{\Delta-1,\Delta}\} \cup \{G_1, G_2\}$, where $G_1$ and $G_2$ are the following two graphs.

![Graph 1](image1.png)

![Graph 2](image2.png)
Upper Bounds

Proof (for $\Delta = 3$ and $g \geq 5$):
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![Graph with labeled vertices and edges]
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\[ \text{Diagram showing a graph with two circles and several connected nodes.} \]
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![Graph diagram]
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![Diagram](image.png)
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![Graph](image-url)
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\begin{center}
\begin{tikzpicture}
  \node[fill] (A) at (0,0) {}; 
  \node[fill] (B) at (1,0) {}; 
  \node[draw] (C) at (2,0) {}; 
  \node[fill] (D) at (3,0) {}; 
  \node[draw] (E) at (4,0) {}; 
  \node[fill] (F) at (5,0) {}; 
  \node[draw] (G) at (6,0) {}; 
  \node[fill] (H) at (7,0) {}; 
  \node[draw] (I) at (8,0) {}; 
  \node[fill] (J) at (9,0) {}; 
  \node[draw] (K) at (10,0) {}; 
  \node[fill] (L) at (11,0) {}; 
  \node[draw] (M) at (12,0) {}; 
  \node[fill] (N) at (13,0) {}; 
  \node[draw] (O) at (14,0) {}; 
  \node[fill] (P) at (15,0) {}; 
  \node[draw] (Q) at (16,0) {}; 
  \node[fill] (R) at (17,0) {}; 
  \node[draw] (S) at (18,0) {}; 
  \node[fill] (T) at (19,0) {}; 
  \node[draw] (U) at (20,0) {}; 
  \node[fill] (V) at (21,0) {}; 
  \node[draw] (W) at (22,0) {}; 
  \node[fill] (X) at (23,0) {}; 
  \node[draw] (Y) at (24,0) {}; 
  \node[fill] (Z) at (25,0) {}; 
  \node[draw] (AA) at (26,0) {}; 

  \draw (A) -- (B);
  \draw (B) -- (C);
  \draw (C) -- (D);
  \draw (D) -- (E);
  \draw (E) -- (F);
  \draw (F) -- (G);
  \draw (G) -- (H);
  \draw (H) -- (I);
  \draw (I) -- (J);
  \draw (J) -- (K);
  \draw (K) -- (L);
  \draw (L) -- (M);
  \draw (M) -- (N);
  \draw (N) -- (O);
  \draw (O) -- (P);
  \draw (P) -- (Q);
  \draw (Q) -- (R);
  \draw (R) -- (S);
  \draw (S) -- (T);
  \draw (T) -- (U);
  \draw (U) -- (V);
  \draw (V) -- (W);
  \draw (W) -- (X);
  \draw (X) -- (Y);
  \draw (Y) -- (Z);

\end{tikzpicture}
\end{center}
Upper Bounds

Proof (for $\Delta = 3$ and $g \geq 5$): We need to find a set $Z_0$ with

$$|\mathcal{F}(Z_0)| \geq 2|Z_0|.$$ 

$\delta(G) \geq 2.$

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Upper Bounds

Conjecture (GR ‘16)

If $G$ is a connected graph of order $n$ and maximum degree 3, then

$$Z(G) \leq \frac{1}{3}n + 2.$$
Upper Bounds

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Theorem (GR ‘16)

If $G$ is a connected graph of order $n$, maximum degree 3, and girth at least 5, then

$$Z(G) \leq \frac{n}{2} - \Omega \left( \frac{n}{\log n} \right).$$
Upper Bounds

Proof (sketch):
Upper Bounds

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![Diagram](image-url)
Upper Bounds

Proof (sketch):
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Upper Bounds

Proof (sketch): If no such subgraph of order $O(\log n)$ exists, then
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Theorem (GR ‘16)

If $G$ is a graph, then

$$Z(G) \leq \sum_{u \in V(G)} \sum_{i=0}^{d_G(u)} (-1)^i \sum_{I \in \binom{N_G(u)}{i}} \left| \{u\} \cup \bigcup_{v \in I} N_G[v] \right|^{-1}.$$
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**Proof:**
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Upper Bounds

Theorem (GR ‘16)

If $G$ is a graph, then

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Proof: Let $u_1, \ldots, u_n$ be a random linear order of the vertices of $G$. Let $Z$ be the set of those vertices $u_i$ such that $u_i$ is not the unique neighbor within $\{u_i, \ldots, u_n\}$ of some vertex $u_j$ with $j < i$. 
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Theorem (GR ‘16)

If $G$ is a graph, then

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Corollary (GR ‘16)

If $G$ is a $r$-regular graph of order $n$ and girth at least 5, then

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$$Z(G) \leq \left( \prod_{i=1}^{r} \left( 1 - \frac{1}{ri + 1} \right) \right) n = \left( 1 - \frac{H_r}{r} \right) n + O \left( \left( \frac{H_r}{r} \right)^2 \right) n.$$

$$H_r = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r} \sim \ln r$$
Lower Bounds

Theorem (AIM group)

*If* $G$ *is a graph, then* $Z(G) \geq \delta(G)$. 

Theorem (Kenter and Davila '14)

Let $G$ be a graph of minimum degree $\delta$ and girth $g$.

(i) If $\delta \geq 3$ and $g \geq 4$, then $Z(G) \geq \delta + 1$.

(ii) If $\delta \geq 2$ and $g \geq 5$, then $Z(G) \geq 2\delta - 2$.

Conjecture (Kenter and Davila '14)

If $\delta \geq 2$ and $g \geq 3$, then $Z(G) \geq (g-2)(\delta-2) + 2$. 
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# Lower Bounds

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## Lower Bounds

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For $g \geq 7$ and $\delta \geq \delta_g$, the conjecture follows using

- $Z(G) \geq tw(G)$ and

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- $Z(G) \geq tw(G)$ and
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**Theorem (GR ‘16)**

*The conjecture holds for $g \in \{4, 5, 6\}$.*
Hereditary Equality with $P(G)$

Recall that $Z(G) \geq P(G)$ with equality for forests and cacti.
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$$\mathcal{ZP} = \{ G : Z(H) = P(H) \text{ for every induced subgraph } H \text{ of } G \}$$
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$\Theta(2, 3, 4)$
Hereditary Equality with $P(G)$

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Folklore

A graph is a cactus if and only if it is $\mathcal{F}$-free for

$$\mathcal{F} = \{ K_4 \} \cup \{ \Theta(\ell_1, \ell_2, \ell_3) : \ell_1, \ell_2, \ell_3 \in \mathbb{N} \text{ and } \ell_2, \ell_3 \geq 2 \}.$$
Hereditary Equality with $P(G)$

**Theorem (GPRS ‘16)**

If $G$ is a graph such that every cycle of $G$ is induced, then the following statements are equivalent.

(i) $G \in \mathcal{ZP}$.

(ii) $G$ is a cactus.

(iii) $G$ is $\mathcal{F}$-free.
The end

Thank you for your attention!