Approximate Polytope Membership Queries

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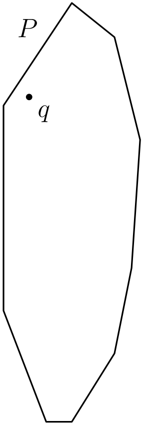
STOC 2011

Polytope Membership Queries

Polytope Membership Queries

Given a polytope P in d-dimensional space, preprocess P to answer membership queries:

Given a point q, is $q \in P$?

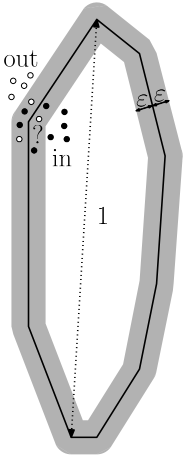


- Assume that dimension d is a constant and P is given as intersection of n halfspaces
- For $d \leq 3$, can be solved with storage $O(n)$ and query time $O(\log n)$ [BCKO10]
- Dual of halfspace emptiness searching

Approximate Polytope Membership Queries

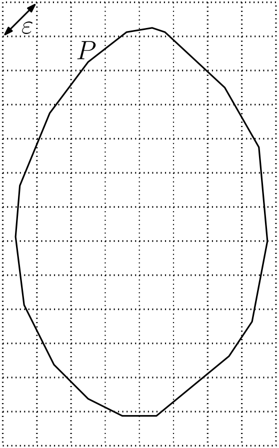
Approximate Version

- An approximation parameter ε is given (at preprocessing time)
- Assume the polytope has diameter 1
- If the query point's distance from P's boundary:
  - $> \varepsilon$: answer must be correct
  - $\leq \varepsilon$: either answer is acceptable



- Polytope approximation is a well studied topic
- We consider the first space-time tradeoffs for the query problem

Bentley *et al.* (Outer) Approximation [BFP82]



- Create a grid with cells of diameter ε
- For each column, store the topmost and bottommost cells intersecting P
- Query processing:
  - Locate the column that contains q
  - Compare q with the two extreme values

Time-Efficient Solution [BFP82]

- $O(1/\varepsilon^{d-1})$ columns
- Storage: $O(1/\varepsilon^{d-1})$
- Query time: $O(1)$ (by integer division)

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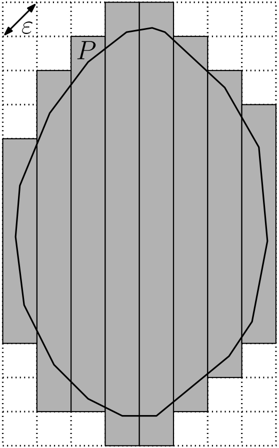

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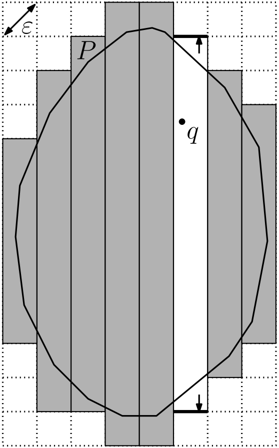

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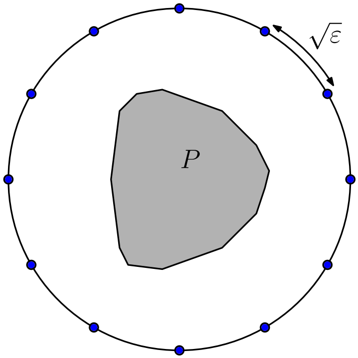

Every unit-diameter polytope can be ε-approximated as the intersection of $O(1/\varepsilon^{(d-1)/2})$ halfspaces [Dud74]

Space-Efficient Solution

Check whether q lies within each Dudley halfspace:

- Storage: $O(1/\varepsilon^{(d-1)/2})$
- Query time: $O(1/\varepsilon^{(d-1)/2})$
- Note: Each halfspace is used to cover a surface patch of size $\sqrt{\varepsilon}$

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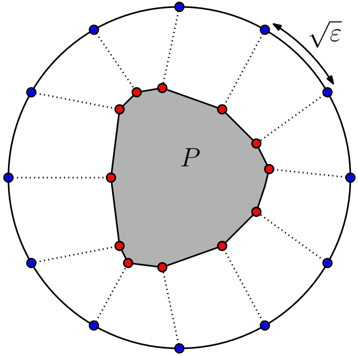

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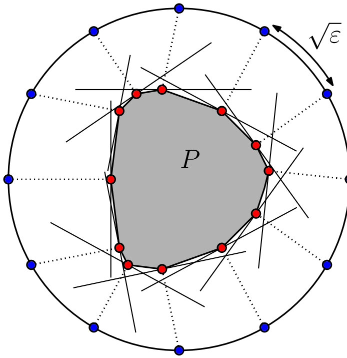

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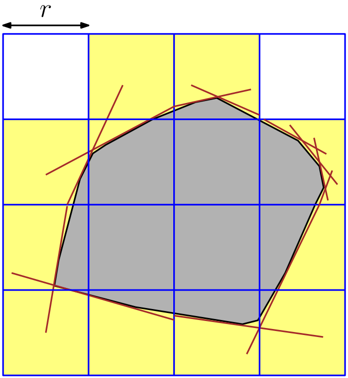
A Simple Tradeoff

- Generate a grid of diameter $r \in [\varepsilon, 1]$
- Preprocessing: For each cell Q intersecting P's boundary:
  - Apply Dudley to $P \cap Q$
  - $O((r/\varepsilon)^{(d-1)/2})$ halfspaces per cell
- Query Processing:
  - Find the cell containing q
  - Check whether q lies within every halfspace for this cell



Tradeoff

- Storage: $O(1/(r\varepsilon)^{(d-1)/2})$
- Query time: $O((r/\varepsilon)^{(d-1)/2})$

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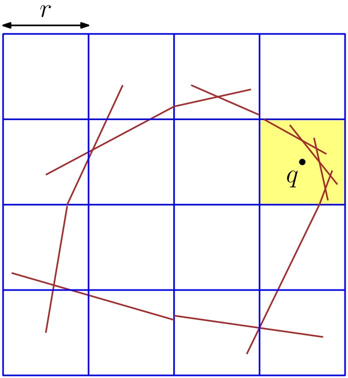

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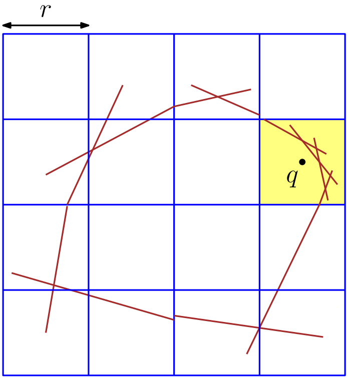

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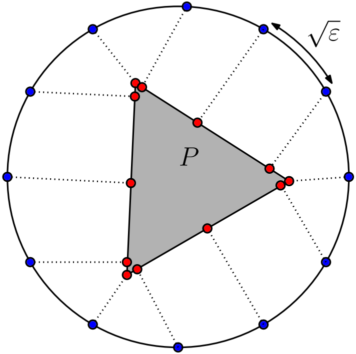
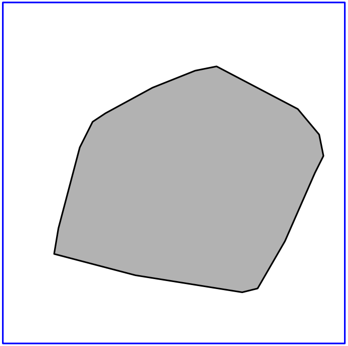
Can we do better? Need a little sensitivity



- Dudley tends to oversample regions of very low and very high curvature
- Finding the smallest number of halfspaces reduces to set cover
- A $\log(1/\varepsilon)$-approximation can be found efficiently (Mitchell and Suri [MS95], Clarkson [Cla93])
- Simple Idea: Recursively subdivide space (quadtree) until the number of approximating halfspaces is small enough

Split-Reduce

$t = 2$



Preprocess:

- Input P, ε, and desired query time t
- $Q \leftarrow$ unit hypercube
- Split-Reduce(Q)

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- Find an ε-approximation of $Q \cap P$
- If at most t facets, then Q stores them
- Otherwise, subdivide Q and recurse

- Query time: $O(\log(1/\varepsilon) + t)$
- Storage: ???

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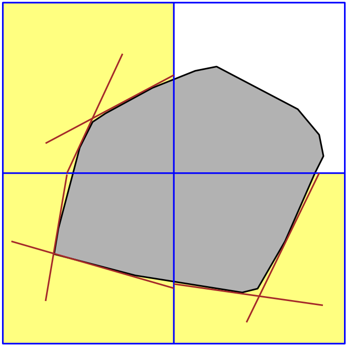

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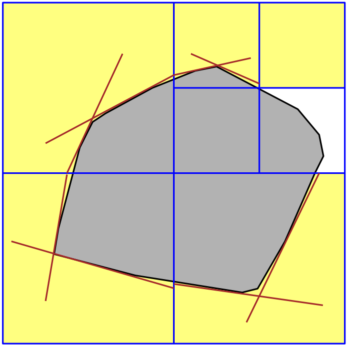

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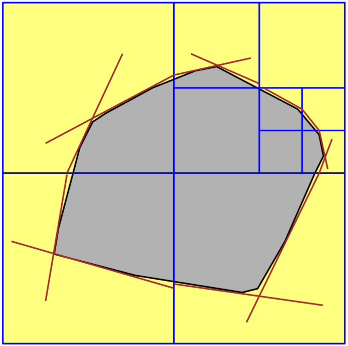

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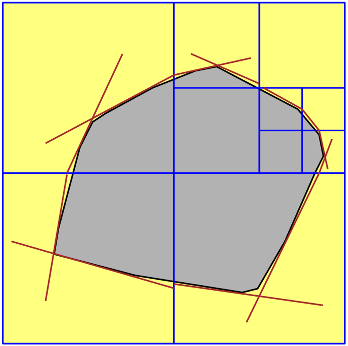

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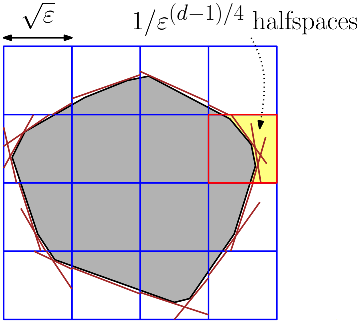
Why it pays to be sensitive

Easy Analysis

Split-Reduce reduces the query time from $1/\varepsilon^{(d-1)/2}$ to $1/\varepsilon^{(d-1)/4}$ with the same $O(1/\varepsilon^{(d-1)/2})$ storage



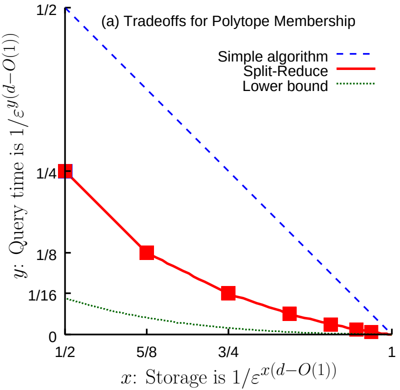
- By Dudley, if diameter $\leq \sqrt{\varepsilon}$, need only $1/\varepsilon^{(d-1)/4}$ halfspaces
  $\Rightarrow$ cells of size $\leq \sqrt{\varepsilon}$ are not subdivided
- Each Dudley halfspace is only needed within a radius of $\sqrt{\varepsilon}$
  $\Rightarrow$ Each halfspace hits only $O(1)$ cells of size $\geq \sqrt{\varepsilon}$
  $\Rightarrow$ The total number of halfspaces needed is $O(1/\varepsilon^{(d-1)/2})$

General Tradeoff

An inductive application of the previous argument yields a space-time tradeoff

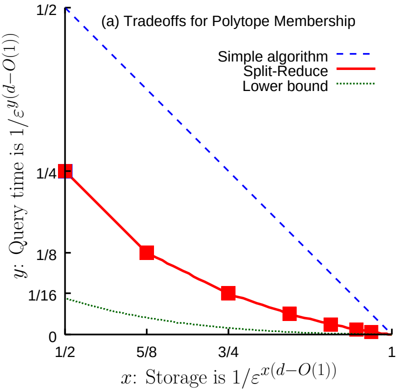
Theorem:

Using Split-Reduce we can answer ε-approximate polytope membership queries with
Storage: $O(1/\varepsilon^{(d-1)/(1-k/2^k)})$
Query time: $O(1/\varepsilon^{(d-1)/2^k})$

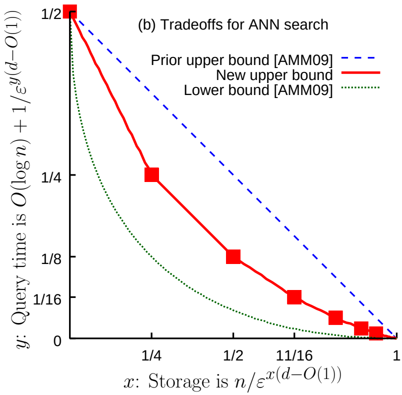


Lower Bound

- The above analysis is not necessarily tight
- We establish a lower bound on Split-Reduce
- The input polytope is a cylinder formed by extruding a $(d-k)$-dimensional ball in k dimensions
- k is chosen to maximize the storage for a given query time



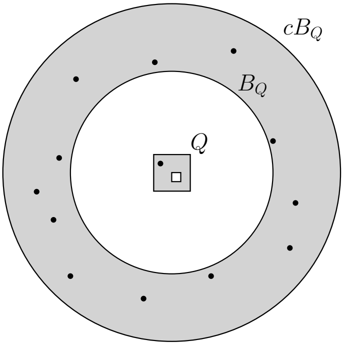
Approximate Nearest Neighbor (ANN) Searching



- ANN: Preprocess n points such that, given a query point q, can find a point within at most $1+\varepsilon$ times the distance to q's nearest neighbor
- Arya, *et al.* [AMM09] gave a solution that is optimal in the extremes of the space-time tradeoff and gave a lower bound
- Our new results improve the tradeoff throughout the middle of the spectrum

Approximate Nearest Neighbor (ANN) Searching

- Arya *et al.* show that it is possible to partition space into cells, each associated with candidates to be the ANN for query points in the cell, such that:
  - Total number of candidates is $\widetilde{O}(n)$
  - All but 1 candidate is inside a constant-radius annulus
- Using lifting we can reduce the search to $\log(1/\varepsilon)$ approximate polytope membership queries



Concluding Remarks

- Improved upper bounds for approximate polytope membership queries
- First space-time tradeoffs
- Simple algorithm – Split-Reduce
- Significant improvements to ANN searching

- Open problem: Tighten the analysis

Thank you!

Bibliography

- [AMM09] S. Arya, T. Malamatos, and D. M. Mount. Space-time tradeoffs for approximate nearest neighbor searching. *J. ACM*, 57:1–54, 2009.
- [BFP82] J. L. Bentley, M. G. Faust, and F. P. Preparata. Approximation algorithms for convex hulls. *Commun. ACM*, 25(1):64–68, 1982.
- [BCKO10] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars. *Computational Geometry: Algorithms and Applications*. Springer, 3rd edition, 2010.
- [Cla93] K. L. Clarkson. Algorithms for polytope covering and approximation. In *Proc. 3rd Workshop Algorithms Data Struct. (WADS)*, pages 246–252, 1993.
- [Dud74] R. M. Dudley. Metric entropy of some classes of sets with differentiable boundaries. *Approx. Theory*, 10(3):227–236, 1974.
- [MS95] J. S. B. Mitchell and S. Suri. Separation and approximation of polyhedral objects. *Comput. Geom.*, 5:95–114, 1995.