On the maximum number of minimum dominating sets minimum total dominating sets and maximum independent sets

Dieter Rautenbach

Universität Ulm

Joint work with Alvarado, Dantas, Henning, and Mohr.

Fricke, Hedetniemi, Hedetniemi, and Hutson '11

> *Does every tree with domination number γ have at most 2^γ minimum dominating sets?*

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Bień at CID 2017

No!

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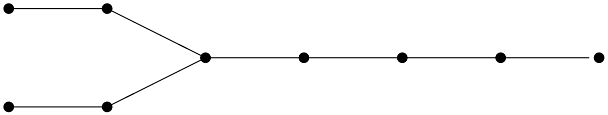

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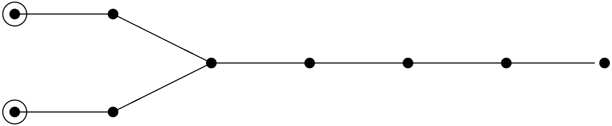

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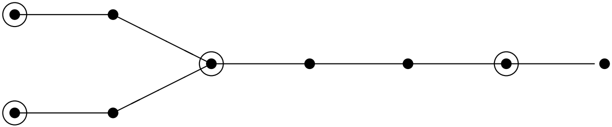

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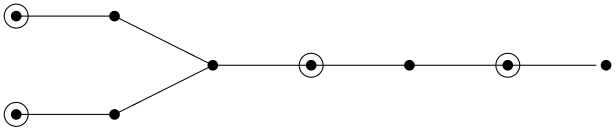

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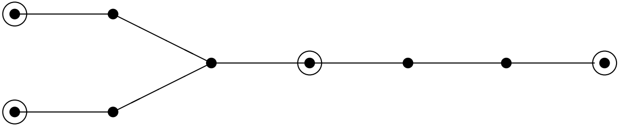

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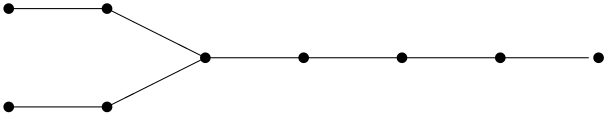

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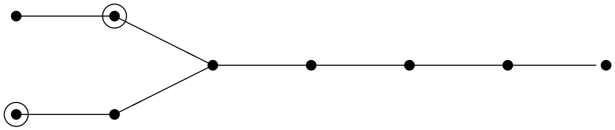

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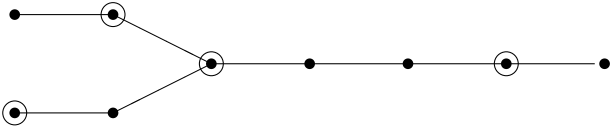

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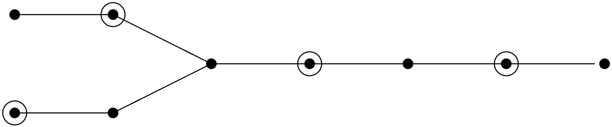

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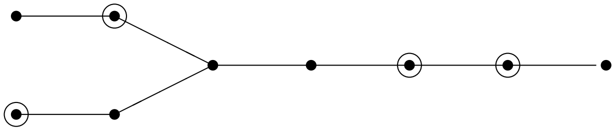

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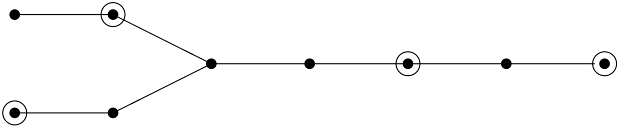

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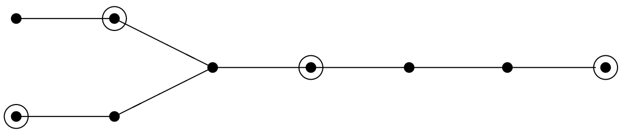

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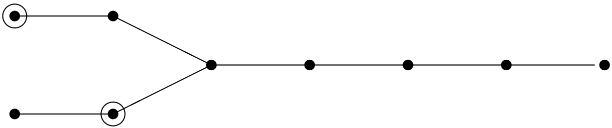

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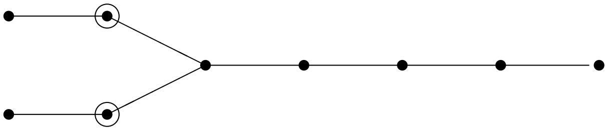

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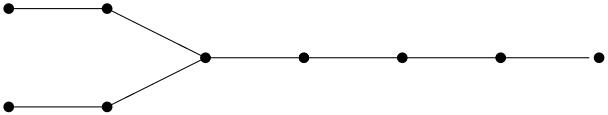

$$3 + 5 + 5 + 5 = 18 > 16 = 2^4$$

Disjoint unions of Bień's tree yield forests with

$$18^{\frac{\gamma}{4}} \approx 2.0598^{\gamma}$$

minimum dominating sets.

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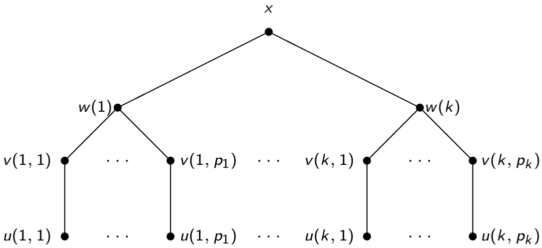
Conjecture (ADMR '18)

A tree with domination number γ has at most

$$O\left(\frac{\gamma 2^{\gamma}}{\ln \gamma}\right)$$

minimum dominating sets.



x
w(1)
w(k)
v(1, 1)
v(1, p1)
v(k, 1)
v(k, pk)
u(1, 1)
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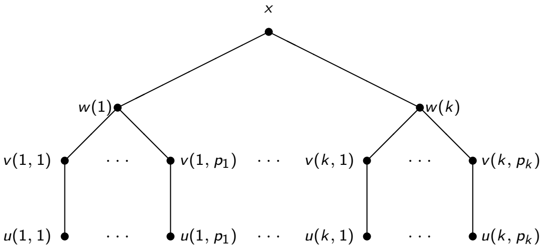
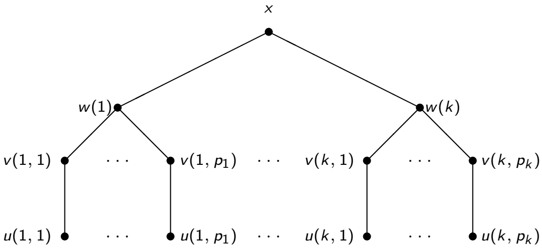


$2^{\gamma-1}$



$$
\begin{aligned}
& 2^{\gamma-1} \\
+ \quad & \big((\gamma-1) \bmod k\big) 2^{\lceil \frac{\gamma-1}{k} \rceil} \left(2^{\lceil \frac{\gamma-1}{k}\rceil}-1\right)^{\left((\gamma-1) \bmod k\right)-1} \left(2^{\lfloor \frac{\gamma-1}{k}\rfloor}-1\right)^{k-\left((\gamma-1)\bmod k\right)} \\
+ \quad & \big(k-((\gamma-1)\bmod k)\big) \left(2^{\lceil\frac{\gamma-1}{k}\rceil}-1\right)^{\left((\gamma-1)\bmod k\right)} 2^{\lfloor \frac{\gamma-1}{k}\rfloor}\left(2^{\lfloor\frac{\gamma-1}{k}\rfloor}-1\right)^{k-\left((\gamma-1)\bmod k\right)-1}
\end{aligned}
$$

Theorem (ADMR '18)

A forest with domination number γ has at most

$$2.4606^{\gamma}$$

minimum dominating sets.

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A forest with domination number γ has at most

$$2.6180^{\gamma}$$

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Proof:

Proof: Let $f(\gamma, s)$ be the maximum number of minimum dominating sets among all forests with domination number γ and s strong support vertices.

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Suppose, for a contradiction,

$$f(\gamma, s) > \alpha^s \beta^{\gamma - s}$$

with γ minimum.

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Consider a longest path $uvw\ldots$ in T.

Case 1

Case 1 v and w are strong support vertices.

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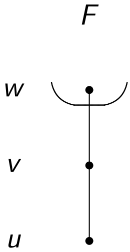

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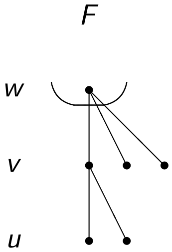

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$f(\gamma, s)$

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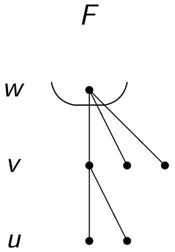

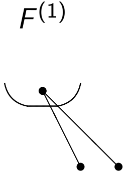

$$f(\gamma, s) \leq f(\gamma - 1, s - 1)$$

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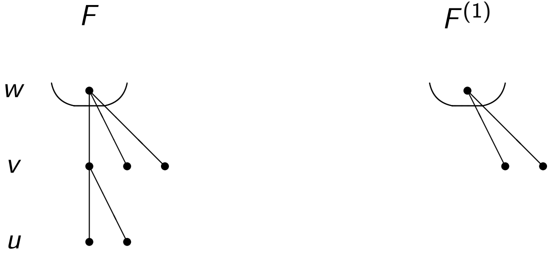

$$f(\gamma, s) \leq f(\gamma - 1, s - 1) \leq \alpha^{s-1}\beta^{\gamma-s}$$

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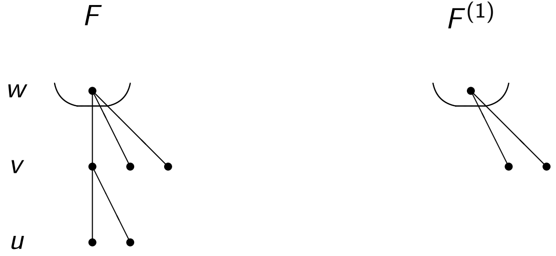

$$f(\gamma, s) \leq f(\gamma - 1, s - 1) \leq \alpha^{s-1}\beta^{\gamma-s} < \alpha^{s}\beta^{\gamma-s}.$$

Case 2

Case 2 w is a strong support vertex but not v.

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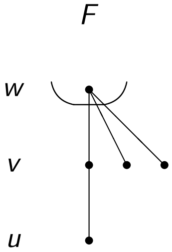

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F $\qquad\qquad$ $F^{(1)}$

$f(\gamma, s)$

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$$f(\gamma, s) \leq 2f(\gamma - 1, s) \leq 2\alpha^s\beta^{\gamma - s - 1}$$

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$$f(\gamma, s) \leq 2f(\gamma-1, s) \leq 2\alpha^s\beta^{\gamma-s-1} \overset{\beta>2}{<} \alpha^s\beta^{\gamma-s}.$$

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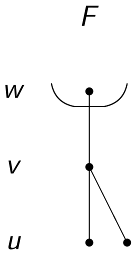

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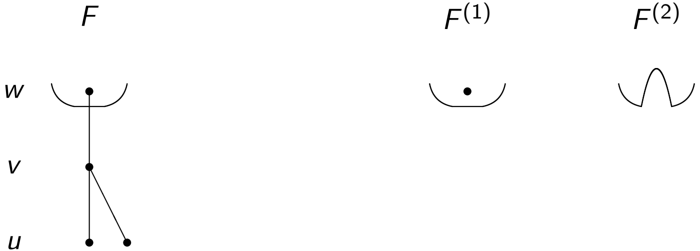

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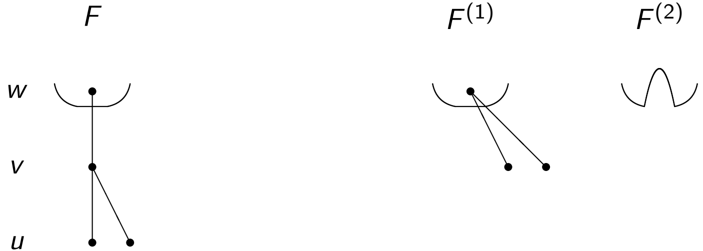

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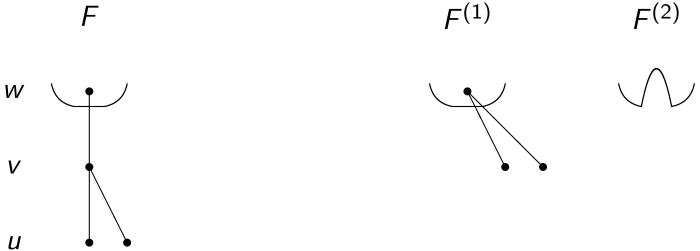

$$f(\gamma, s) \quad \leq$$

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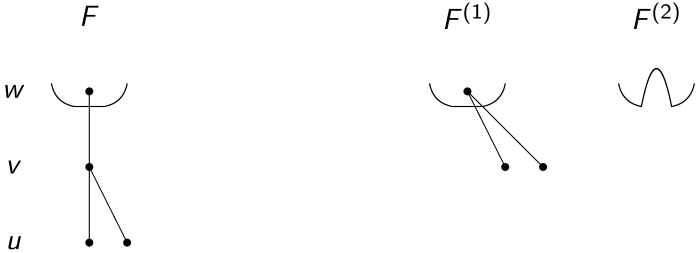

$$f(\gamma, s) \quad \leq \quad f\left(\gamma-1, s\right)+f\left(\gamma-1, s-1\right)$$

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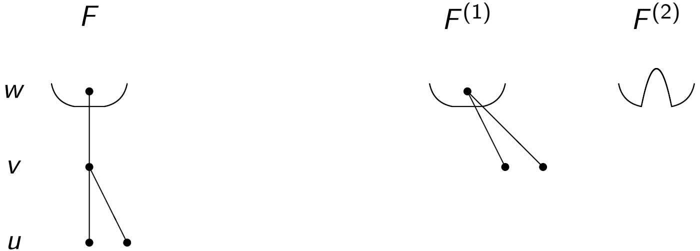

$$\begin{aligned} f(\gamma, s) &\leq f(\gamma-1, s) + f(\gamma-1, s-1) \\ &\leq \alpha^{s}\beta^{\gamma-s-1} + \alpha^{s-1}\beta^{\gamma-s} \end{aligned}$$

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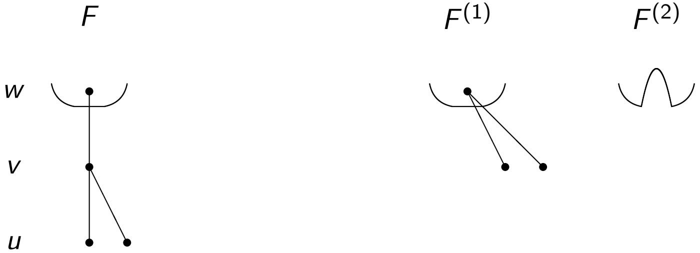

$$
\begin{aligned}
f(\gamma, s) &\leq f(\gamma-1, s) + f(\gamma-1, s-1) \\
&\leq \alpha^s \beta^{\gamma-s-1} + \alpha^{s-1}\beta^{\gamma-s} \\
&= \left(\frac{1}{\beta} + \frac{1}{\alpha}\right)\alpha^s \beta^{\gamma-s}
\end{aligned}
$$

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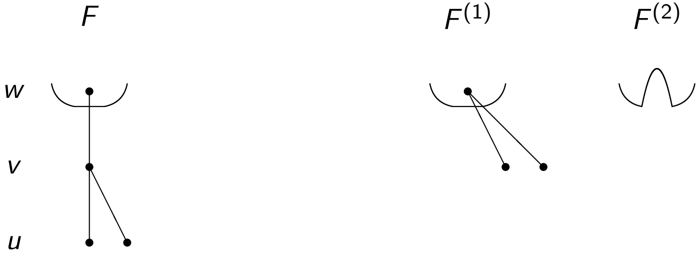

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&= \alpha^{s}\beta^{\gamma-s},
\end{aligned}
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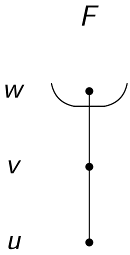

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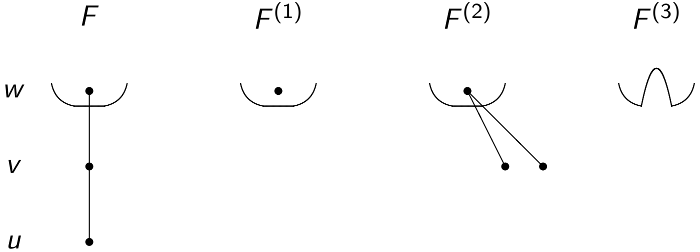

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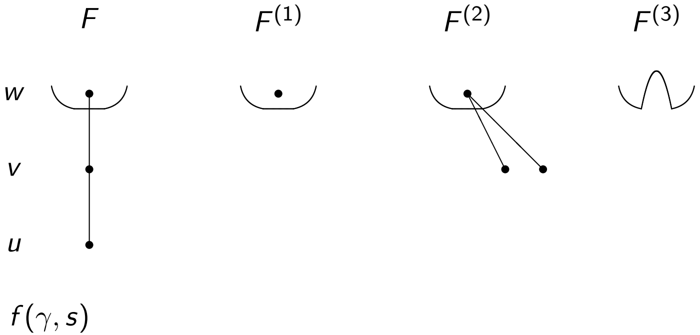

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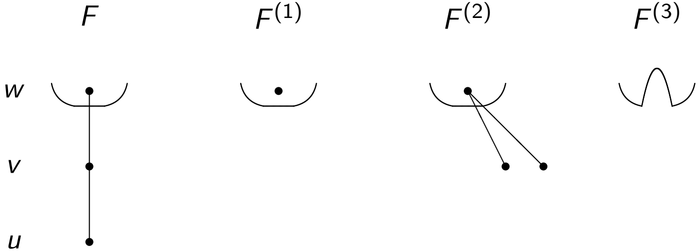

$$f(\gamma, s) \quad \leq \quad f\left(\gamma - 1, s\right) + f\left(\gamma - 1, s + 1\right) + f\left(\gamma - 1, s\right)$$

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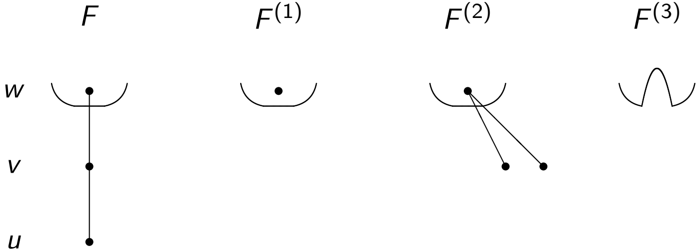

$$\begin{aligned} f(\gamma, s) &\leq f(\gamma-1, s) + f(\gamma-1, s+1) + f(\gamma-1, s) \\ &\leq \alpha^{s}\beta^{\gamma-s-1} + \alpha^{s+1}\beta^{\gamma-s-2} + \alpha^{s}\beta^{\gamma-s-1} \end{aligned}$$

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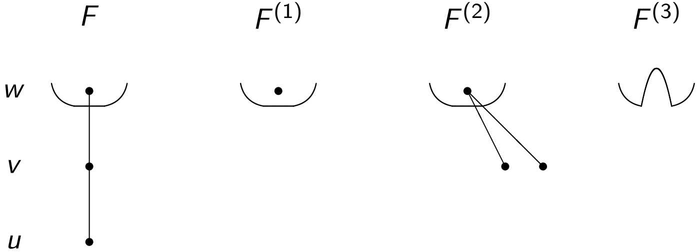

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&\leq \alpha^{s}\beta^{\gamma-s-1} + \alpha^{s+1}\beta^{\gamma-s-2} + \alpha^{s}\beta^{\gamma-s-1} \\
&= \left(\frac{2}{\beta} + \frac{\alpha}{\beta^{2}}\right)\alpha^{s}\beta^{\gamma-s}
\end{aligned}
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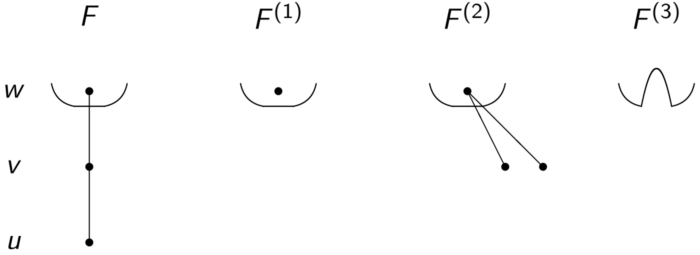
Case 4 v and w are both no strong support vertices.



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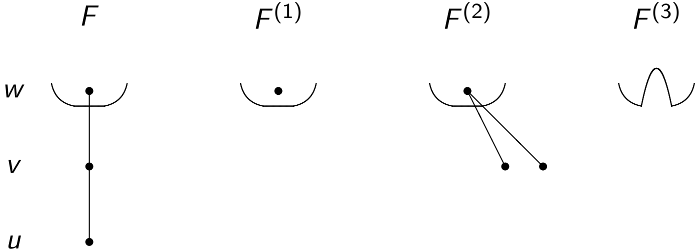

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□

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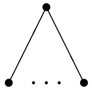

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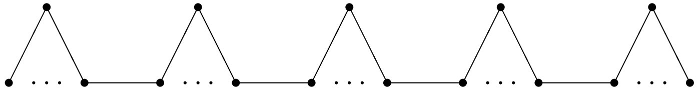

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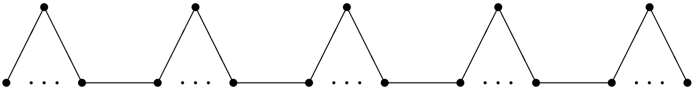

Conjecture (HMR '18)

If a tree T has order n at least 2 and total domination number γ_t, then T has at most

$$\left(\frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}$$

minimum total dominating sets.

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$$\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}+o\left(\frac{n}{\gamma_t}\right)}$$

Proof sketch:

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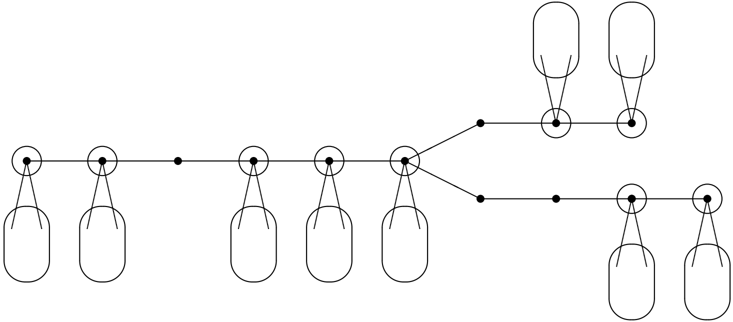

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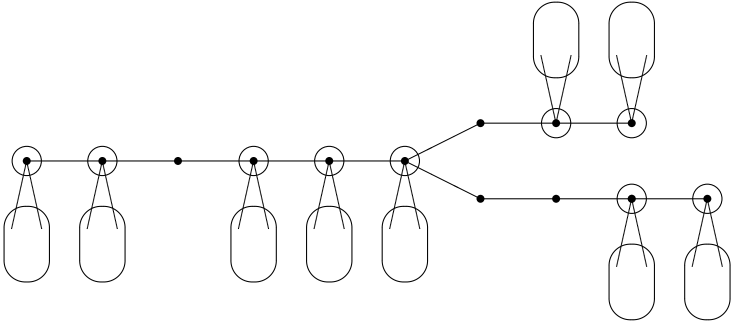

$n(T') \leq 2\gamma_t$.

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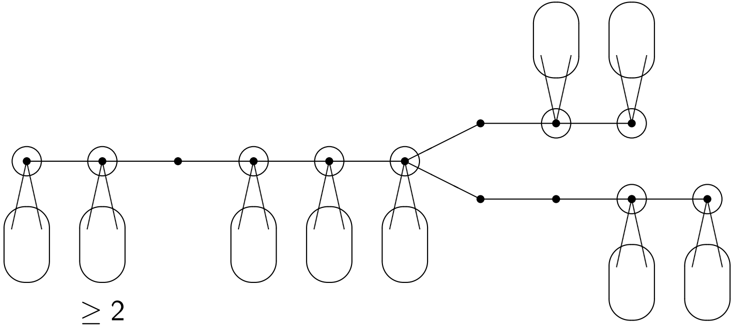

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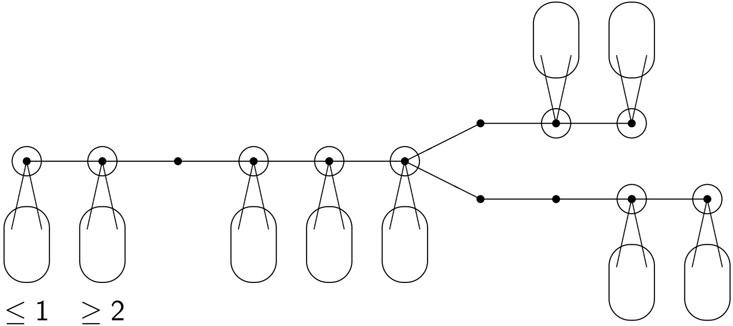

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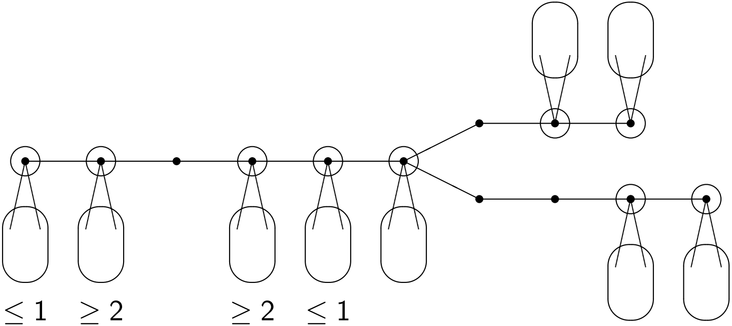

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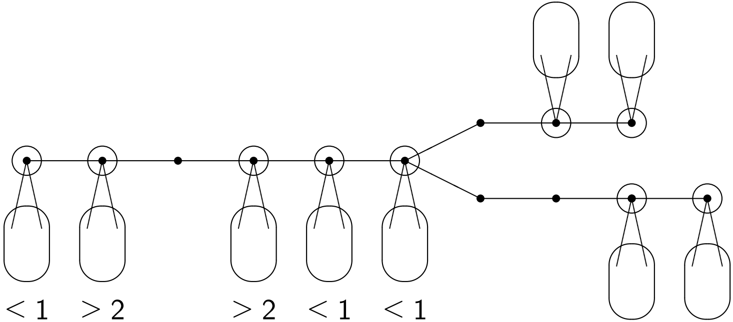

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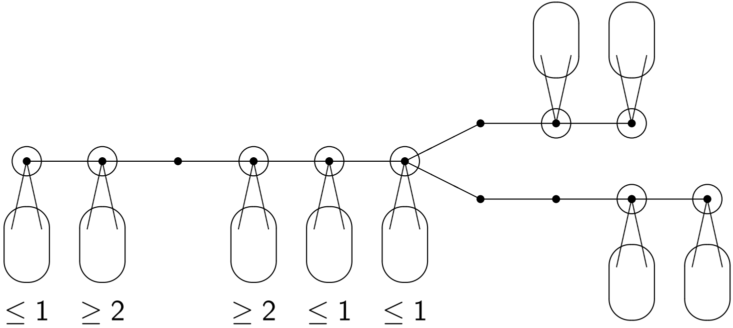

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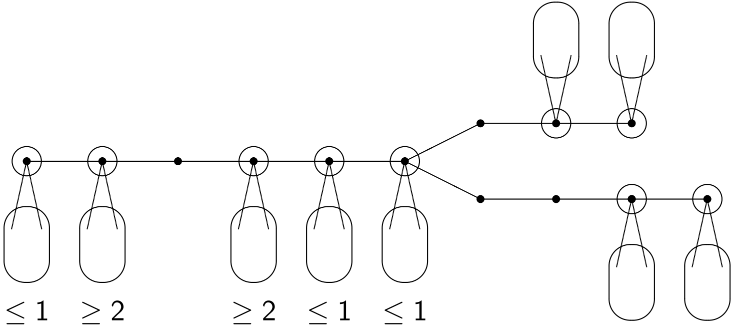

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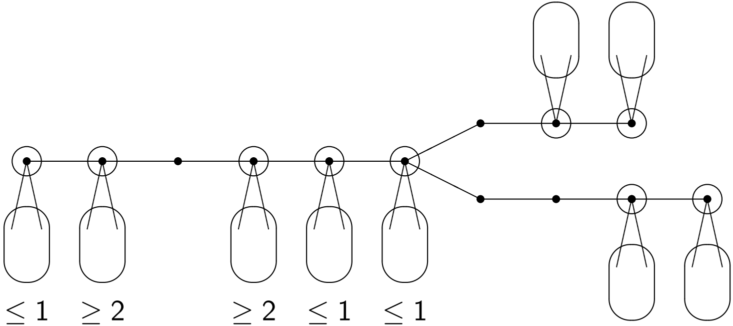

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What about independence?

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Theorem (MR '18)

If T is a tree of order n and independence number α, then T has at most

$$\begin{cases} 2^{n-\alpha-1}+1 & \text{, if } 2\alpha = n \text{, and} \\ 2^{n-\alpha-1} & \text{, if } 2\alpha > n \end{cases}$$

maximum independent sets.
Furthermore, equality holds if and only if T arises by subdividing $n-\alpha-1$ edges of $K_{1,\alpha}$ once.

Let $T_p(n)$ be the Turán graph of order n with p partite sets.

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For $n, p \in \mathbb{N}$, the graph $T_{p-1}(n)$ maximizes $m(G)$ among all graphs G of order n with no p-clique.

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be the number of q-cliques in G.

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If G is a graph of order n with no p-clique, then

$$\sharp\omega^{(q)}(G) \leq \sharp\omega^{(q)}\left(T_{p-1}(n)\right)$$

with equality if and only if $G = T_{p-1}(n)$.

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Proof: Let G with order n and no p-clique maximize $\sharp\omega^{(q)}(G)$.

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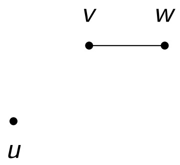

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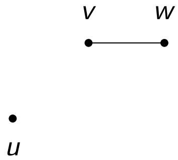

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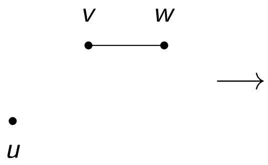

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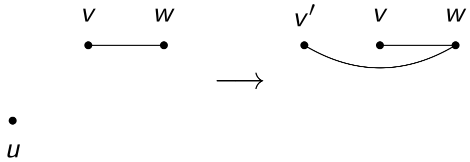

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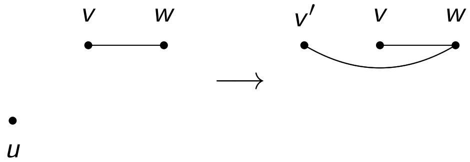

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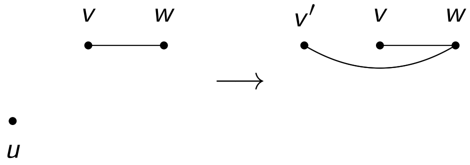

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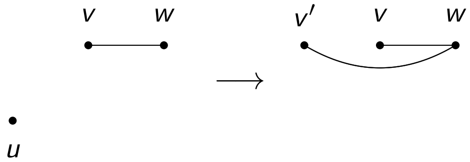

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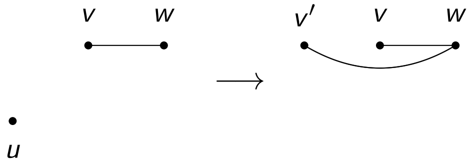

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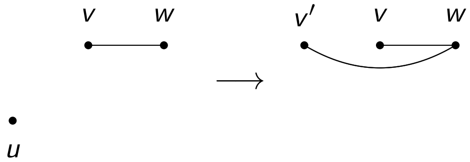

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q-cliques.

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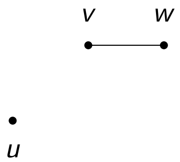

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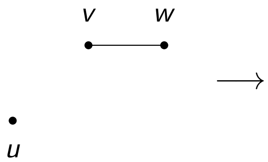

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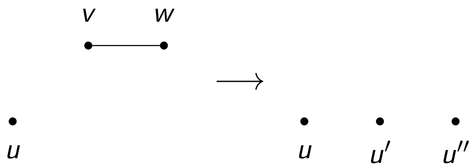

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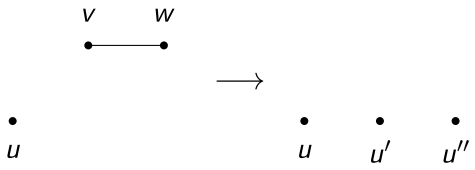

Since vw belongs to some q-clique, $G_0 - v - w + u' + u''$ has

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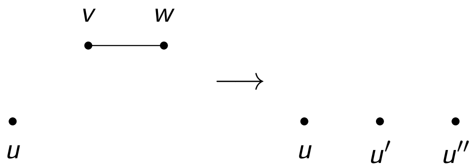

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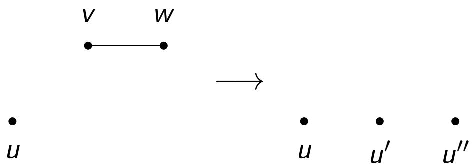

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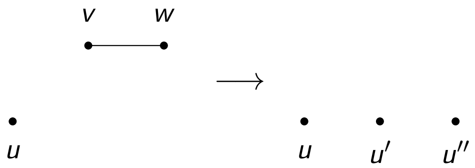

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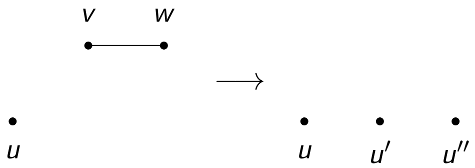

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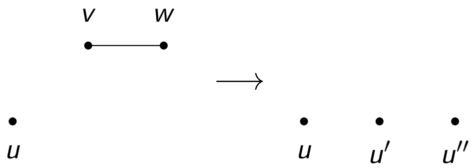

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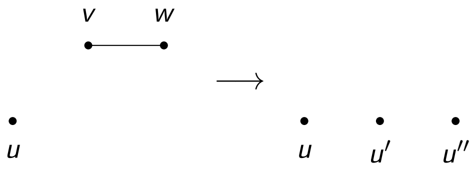

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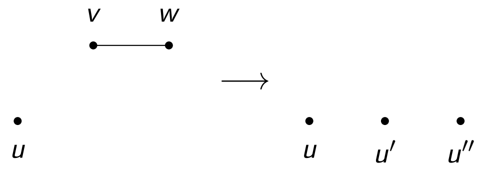

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$$\sharp\alpha(G) \leq g(n,\alpha) = \left\lceil \frac{n}{\alpha} \right\rceil^{n \bmod \alpha} \left\lfloor \frac{n}{\alpha} \right\rfloor^{\alpha-(n \bmod \alpha)}.$$

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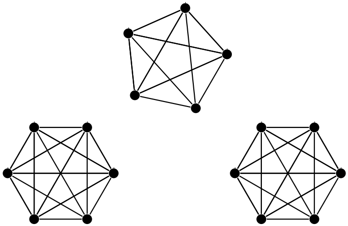

Figure: The graph $G(17,3) = \overline{T}_3(17)$

Moon and Moser's result ('65) on the maximum number of maximal independent sets implies

$$\sharp\alpha(G) \leq \begin{cases} 3^{\frac{n}{3}} & \text{, if } n \mod 3 = 0, \\ 4 \cdot 3^{\frac{n-4}{3}} & \text{, if } n \mod 3 = 1, \text{ and} \\ 2 \cdot 3^{\frac{n-2}{3}} & \text{, if } n \mod 3 = 2, \end{cases}$$

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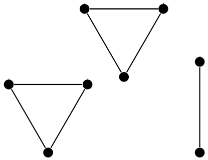

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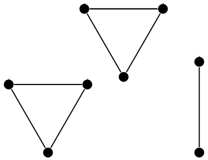

Griggs, Grinstead, and Guichard have shown a similar result for connected graphs.

What happens with $\sharp\alpha(G)$ in connected graphs?

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$$\max\Big\{ \quad \sharp\alpha(G): \quad G \text{ is a connected graph of order } n \text{ and independence number } \alpha\Big\},$$

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which they verify for $\alpha \in \{1, 2, n-3, n-2, n-1\}$.

Definition

For $\frac{n}{\alpha} \geq 2$ define $F(n, \alpha)$ as the graph with α cliques, $V(G) = C_0 \dot{\cup} \ldots \dot{\cup} C_{\alpha-1}$ of order $\left\lceil \frac{n}{\alpha} \right\rceil$ and $\left\lfloor \frac{n}{\alpha} \right\rfloor$. The only other edges of $F(n, \alpha)$ are incident to a vertex x_0 of the largest clique that has exactly one neighbor in every other clique.

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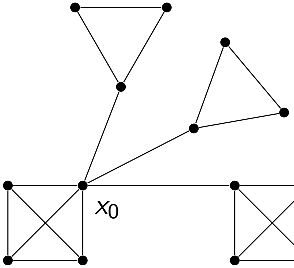

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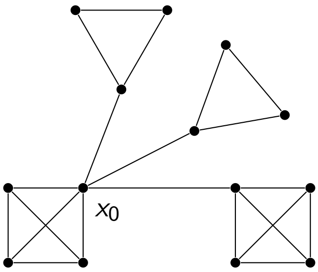
$$\mathcal{F}(n, \alpha) = \begin{cases} \{F(n, \alpha), C_5\} & \text{if } n = 5 \text{ and } \alpha = 2 \\ \{F(n, \alpha)\} & \text{otherwise.} \end{cases}$$



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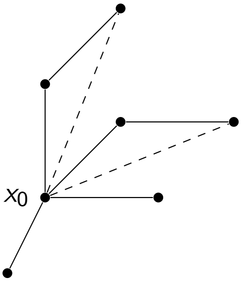

Figure: A member of $\mathcal{F}(7, 4)$, the dashed lines stand for possible edges

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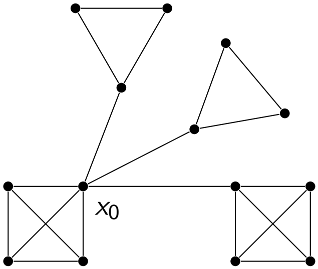

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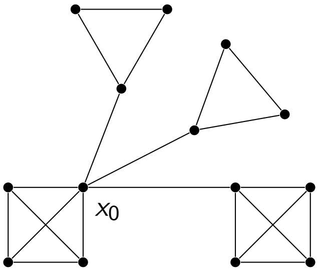

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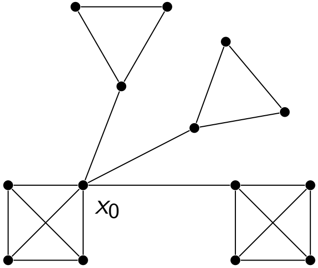

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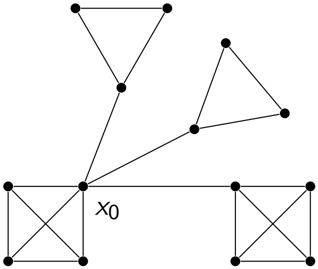

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$$f(n, \alpha) \;=\; g(n-1, \alpha) + \left(\left\lfloor \frac{n}{\alpha} \right\rfloor - 1\right)^{\alpha - n \bmod \alpha} \cdot \left(\left\lceil \frac{n}{\alpha} \right\rceil - 1\right)^{n \bmod \alpha - 1}$$

Theorem

If G is a connected graph of order n and independence number α with $\alpha < n$, then $\sharp\alpha(G) \leq f(n,\alpha)$ with equality if and only if G is isomorphic to a graph in $\mathcal{F}(n,\alpha)$.

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Let the vertex x of G_0 maximize $\sharp\alpha(G_0, x)$. Define $N := N_{G_0}[x]$.

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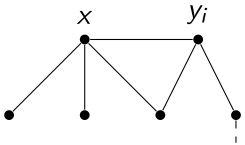

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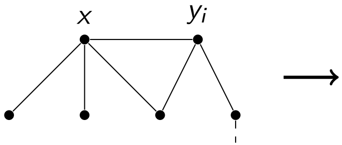

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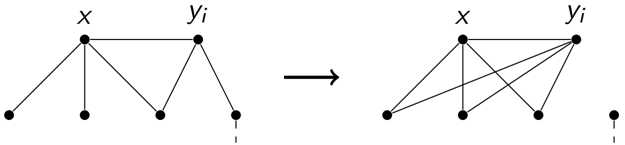

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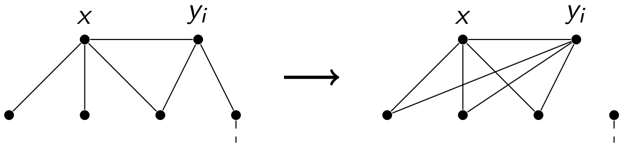

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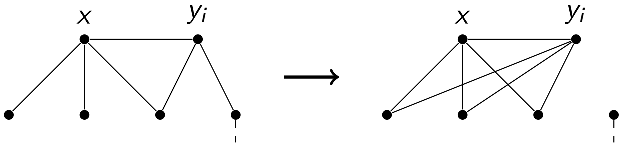

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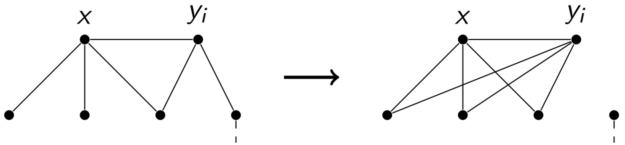

For the number of maximum independent sets in G_i it holds that:

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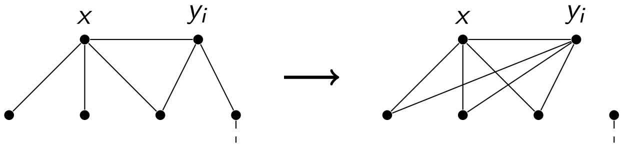

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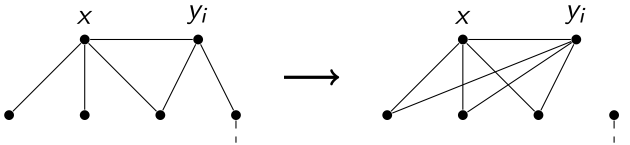

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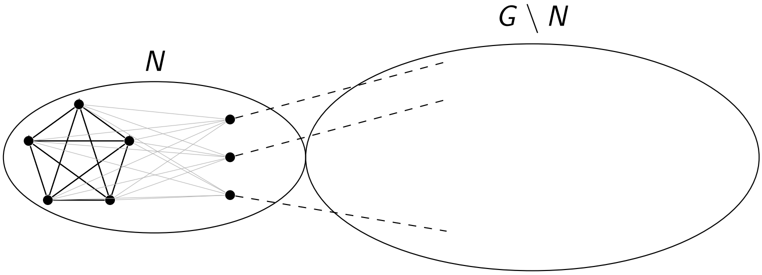
Claim

G is isomorphic to a graph in $\mathcal{F}(n, \alpha)$.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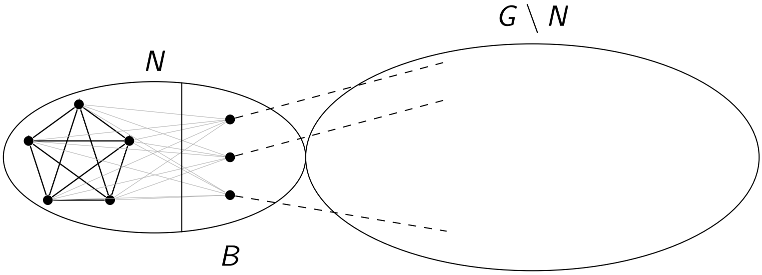

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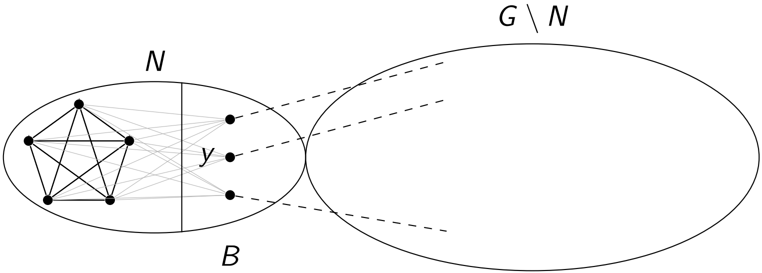

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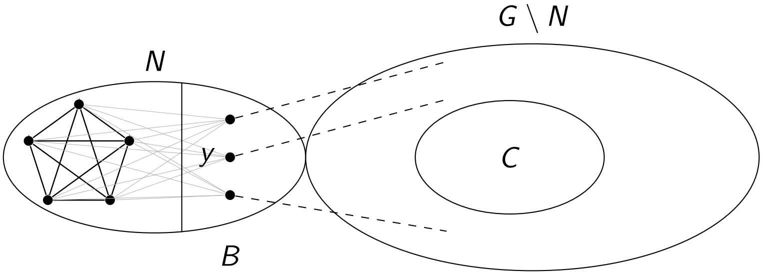

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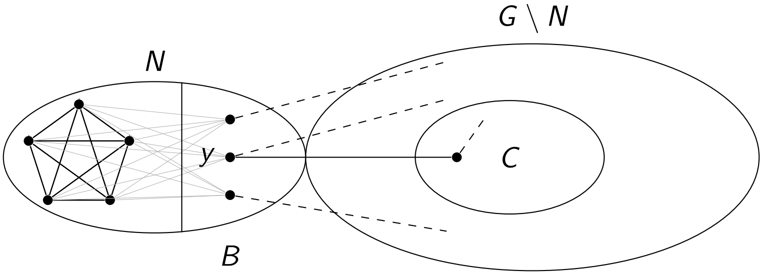

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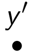
There is a vertex y in B, and a private component C of y such that C has order at least 2, and y has exactly one neighbor in $V(C)$.



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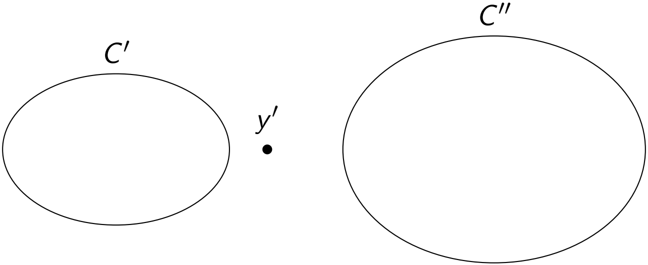

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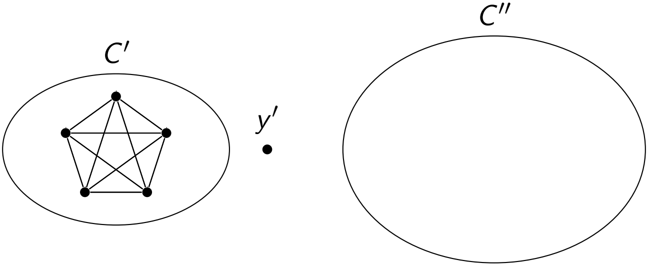

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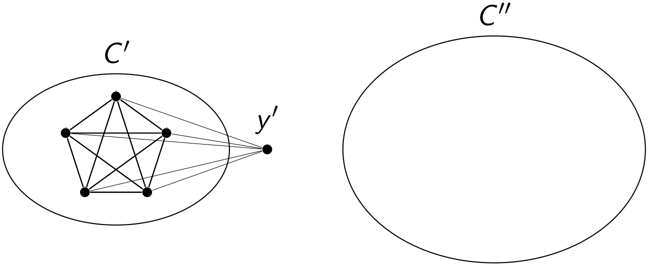

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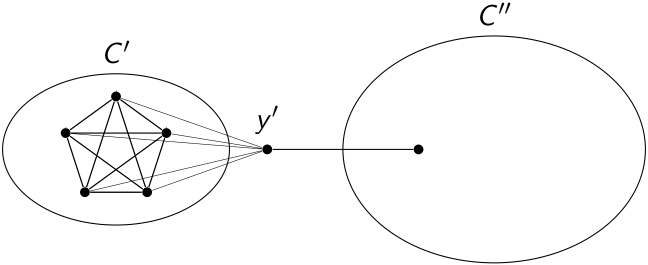

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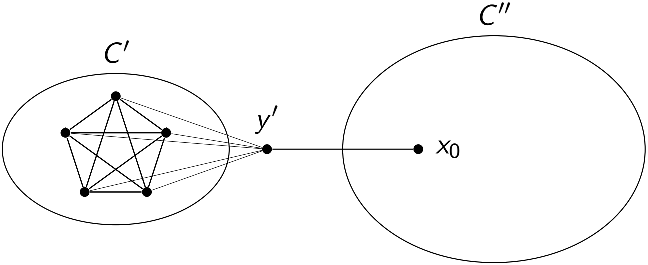

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Thank you!