Misère Geography and Vertex NimG are PSPACE-hard

Gabriel Renault

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Impartial combinatorial games

An impartial combinatorial game is a game:

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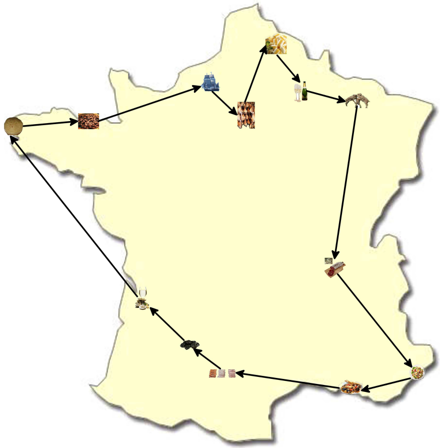
We consider two winning conventions:

- in normal version, the player who plays the last move wins
- in misère version, the player who plays the last move loses

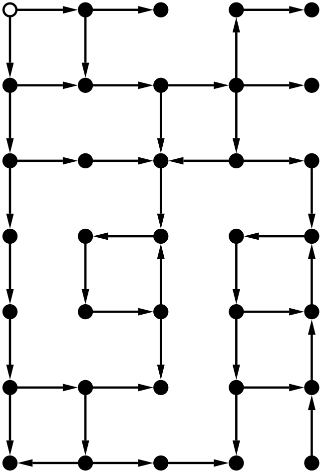
Geography



A directed graph G with a token on a vertex.

A move is to slide the token to an out-neighbour and delete the previous vertex.

The game can also be played on an undirected graph.



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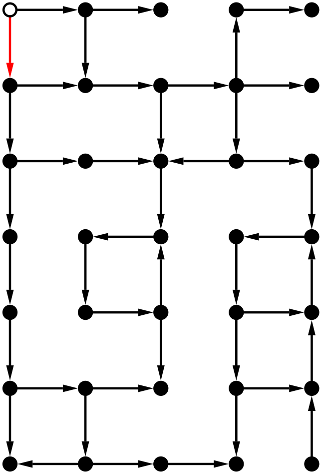

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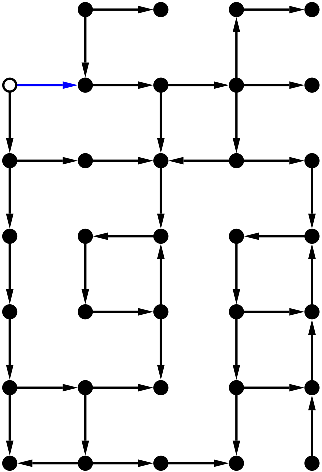

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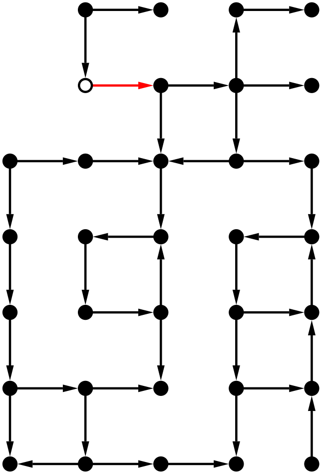

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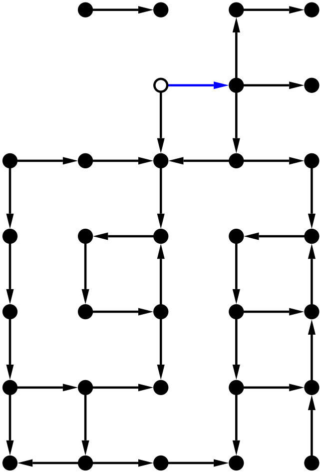

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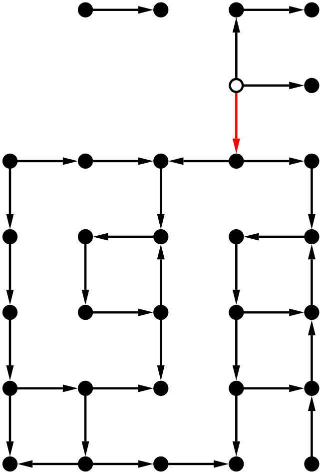

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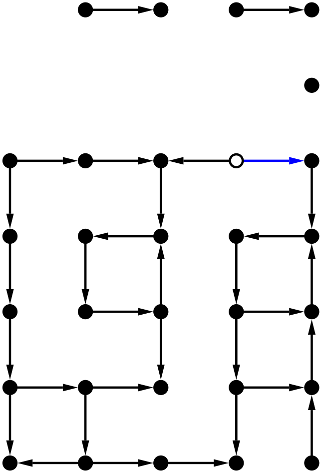

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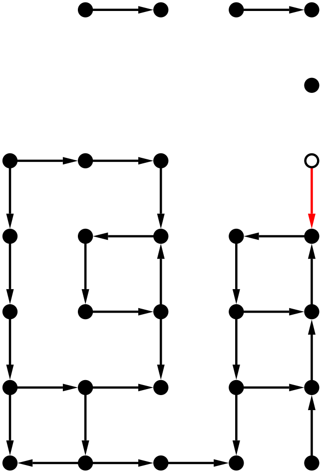

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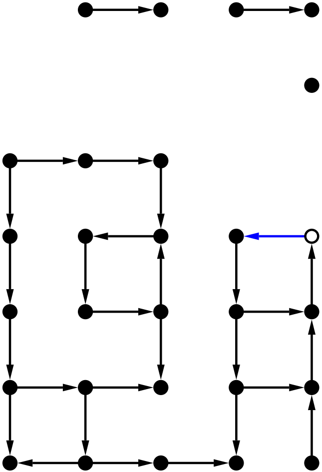

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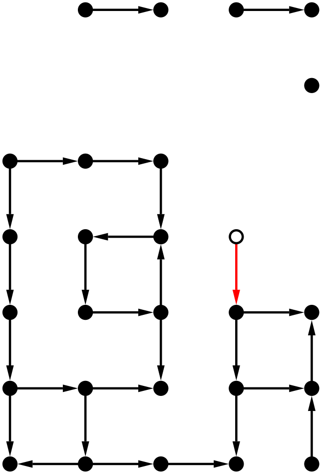

GEOGRAPHY

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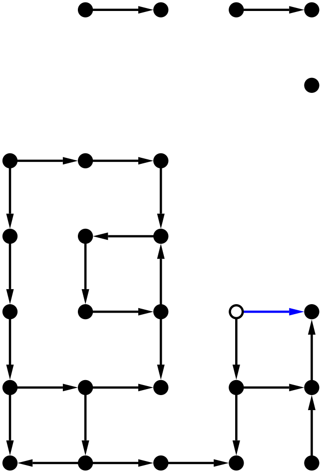

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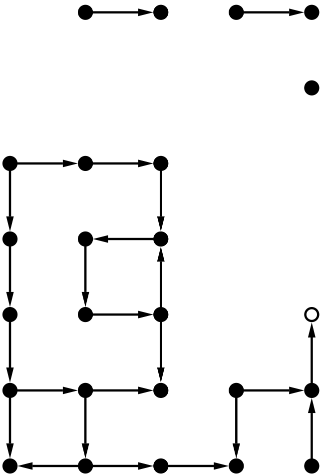

Complexity of GEOGRAPHY

Under the normal convention:

- On directed graphs, the problem is PSPACE-complete, even when played on planar bipartite graphs with maximum degree 3. (Lichtenstein, Sipser, 1979)
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Under the misère convention?

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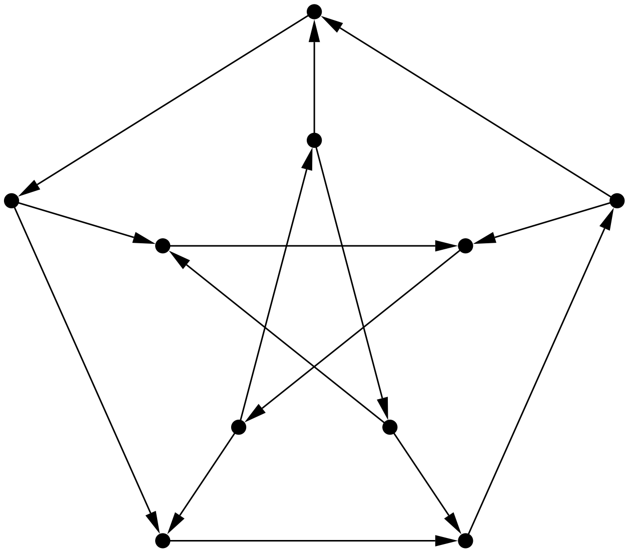

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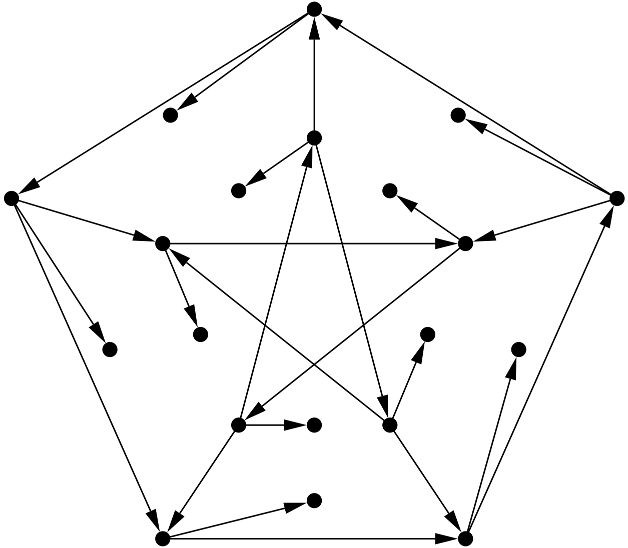
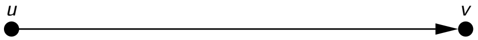
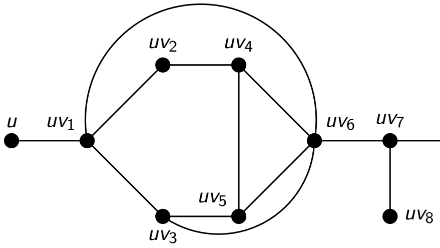


The gadgets





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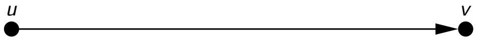

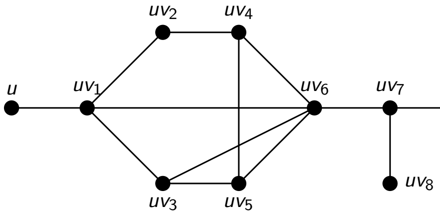

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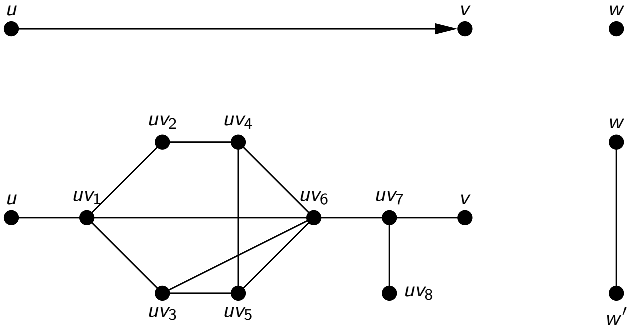

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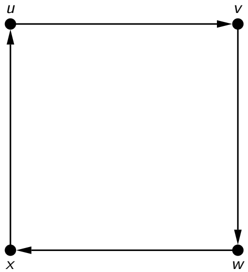

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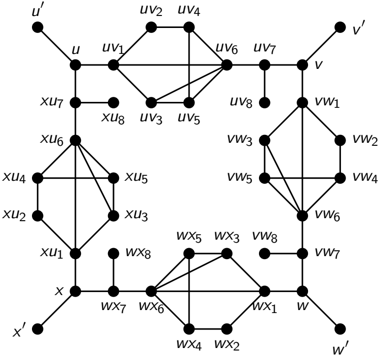

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uv_2
uv_4
v'
u
uv_1
uv_6
uv_7
v
xu_7
xu_8
uv_3
uv_5
uv_8
vw_1
xu_6
vw_3
vw_2
xu_4
xu_5
vw_5
vw_4
xu_2
xu_3
vw_6
wx_5
wx_3
vw_8
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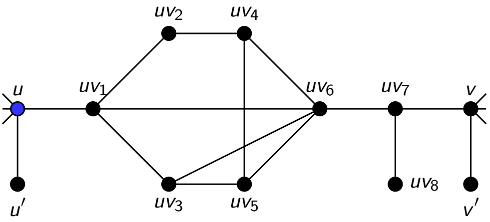
Losing moves are losing



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- A move from u to u' is losing.
- A move from v to uv_7 is losing.
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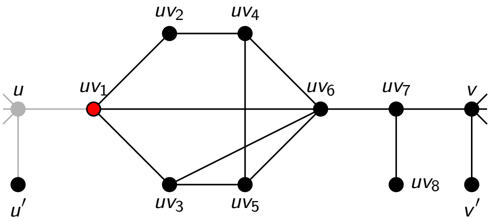

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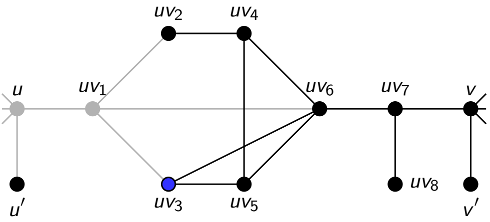

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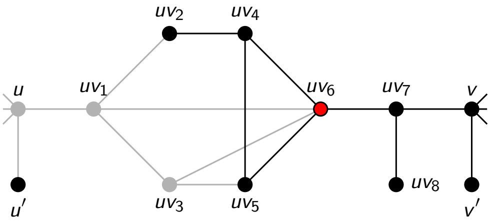

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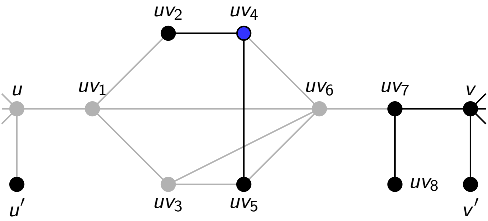

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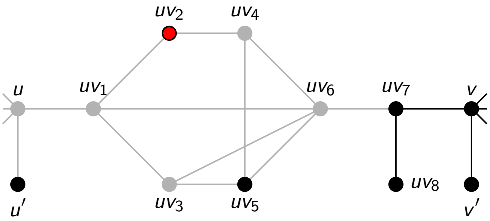

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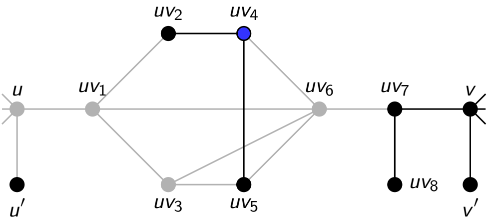

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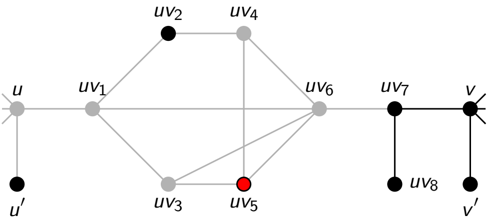

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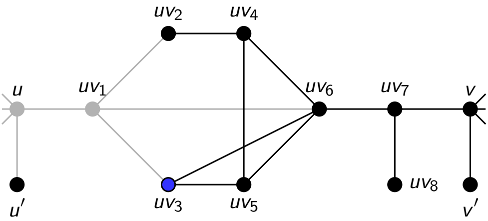

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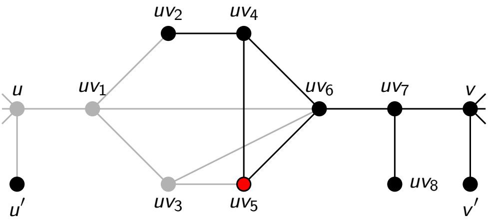

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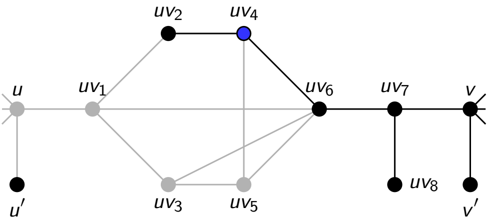

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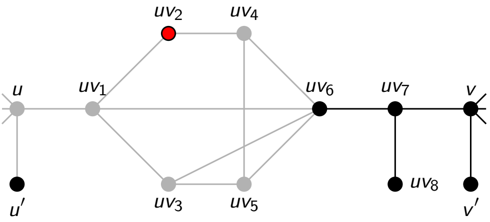

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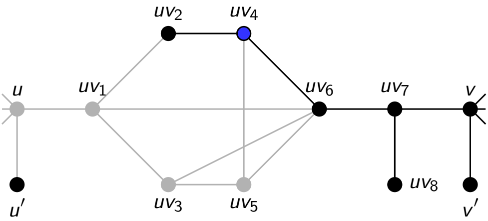

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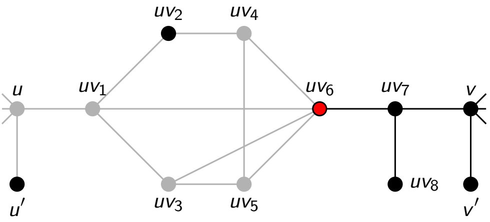

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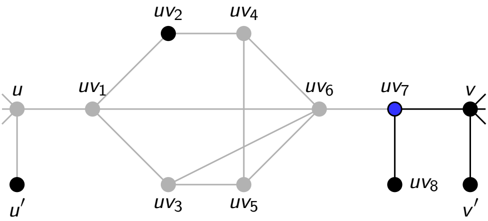

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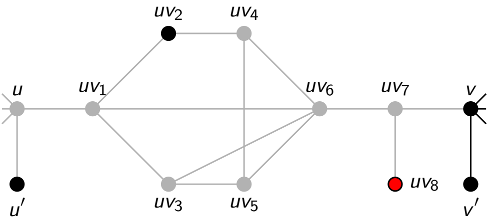

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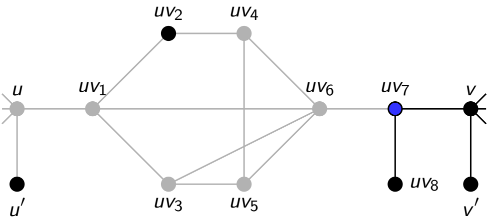

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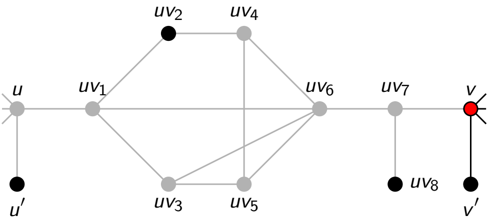

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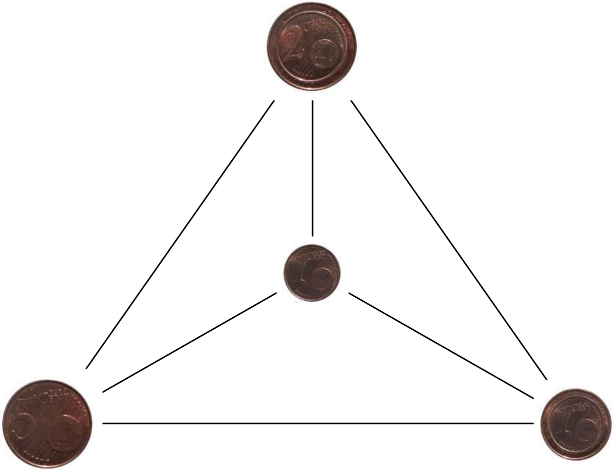
Under the misère convention:

- On directed graphs, the problem is PSPACE-complete, even when played on planar bipartite graphs with maximum degree 4.
- On undirected graphs, the problem is PSPACE-complete, even when played on planar graphs with maximum degree 5.

Vertex NimG

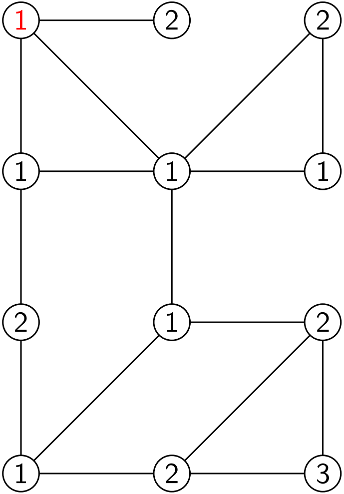


Vertex NimG-MR

A weighted graph G with a token on a vertex.

A move is to slide the token to an out-neighbour and remove some weight from this out-neighbour.

All weights must remain non-negative integers.



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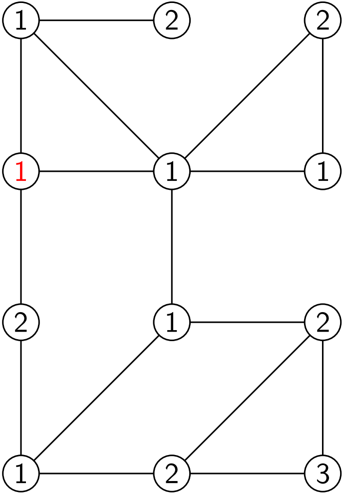

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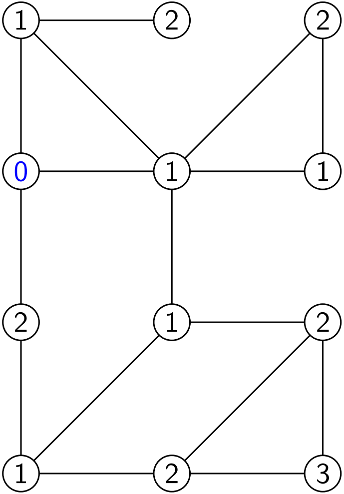

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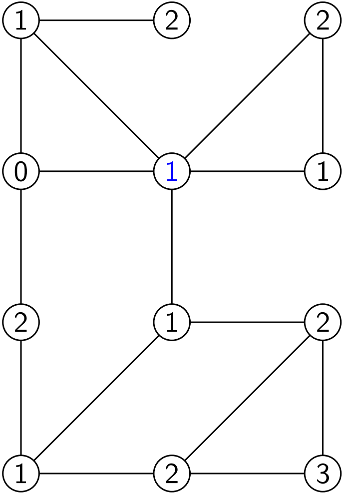

Vertex NimG-MR

A weighted graph G with a token on a vertex.

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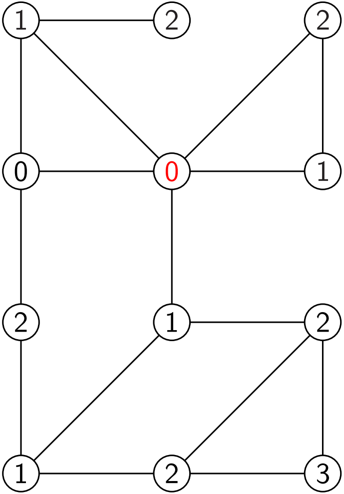

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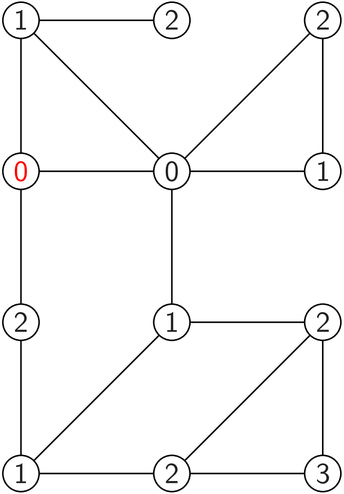

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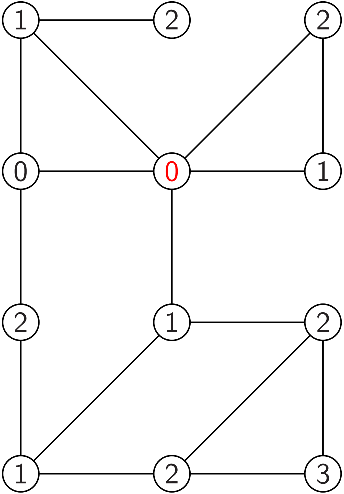

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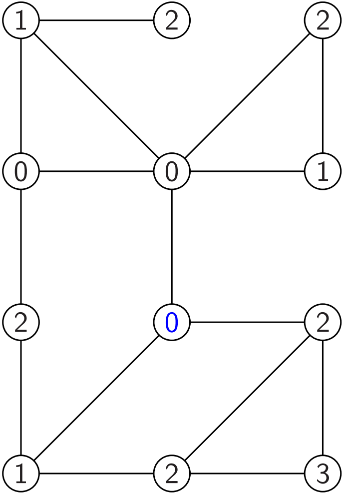

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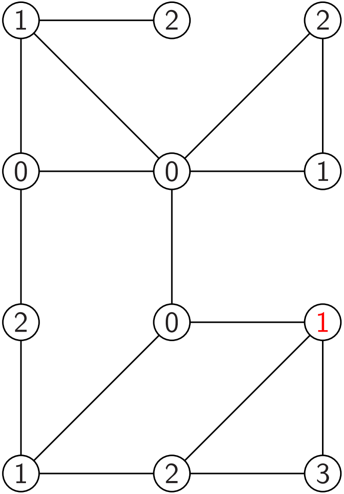

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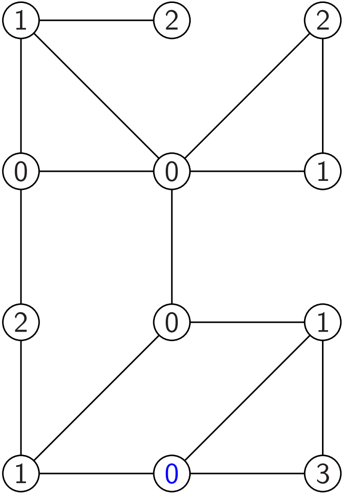

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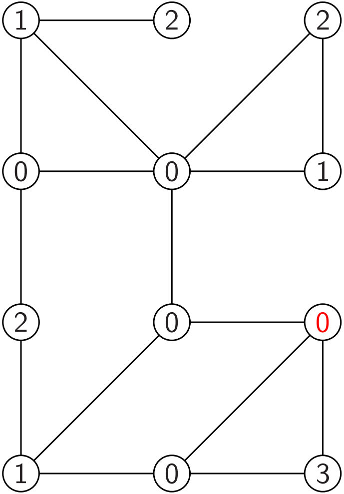

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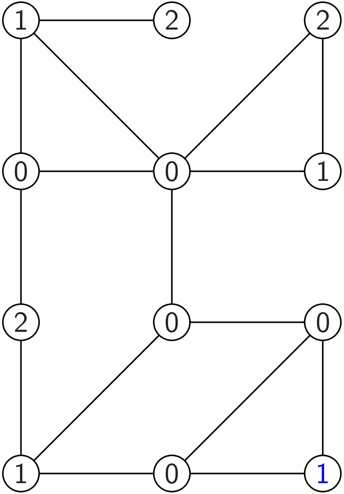

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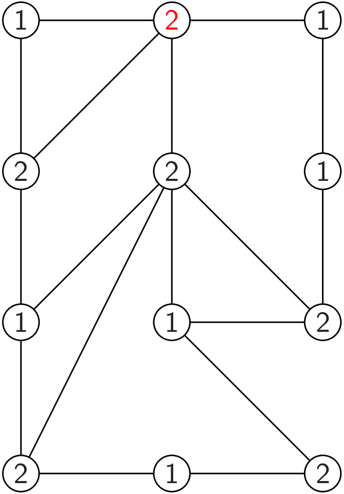

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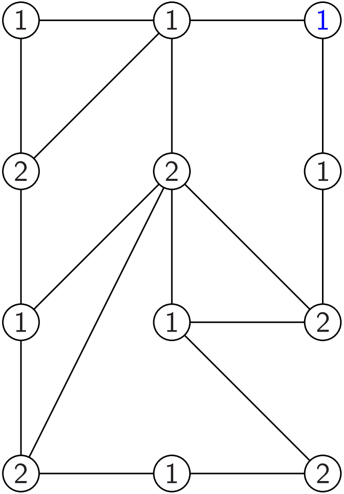

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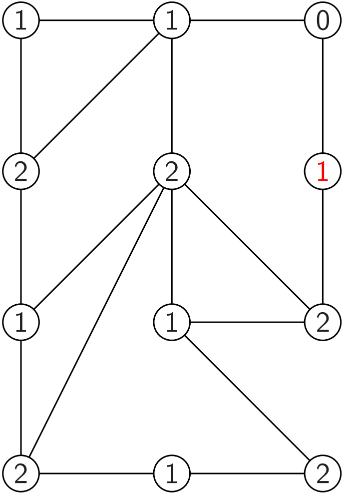

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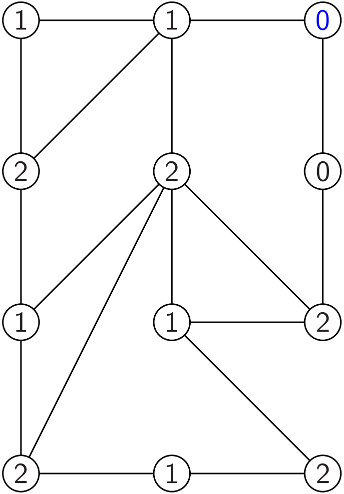

Complexity of Vertex NimG

Under the normal convention:

- On the MR version, the problem is PSPACE-hard, even when played on planar graphs with maximum degree 3.
  (Burke, George, 2011)
- On the RM version, the problem is polynomial.
  (Duchêne, R., 2014)

Complexity of VERTEX NIMG

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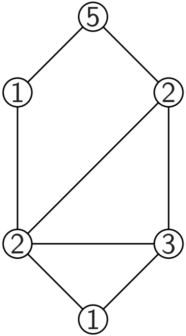
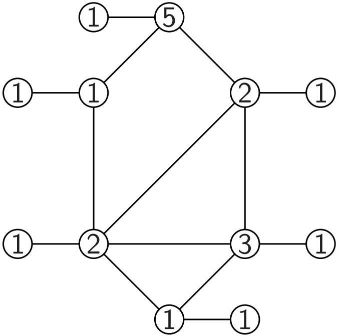
Under the misère convention?

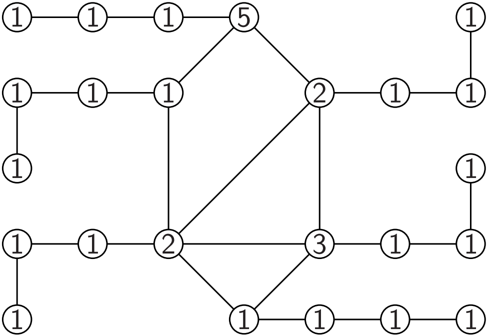
MR version



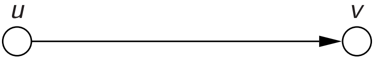
MR version

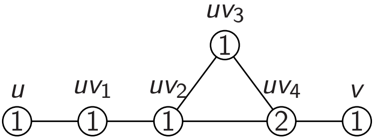


MR version

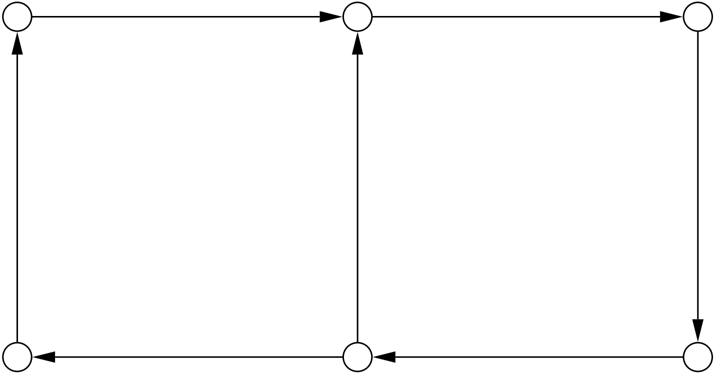


RM Gadgets





Example



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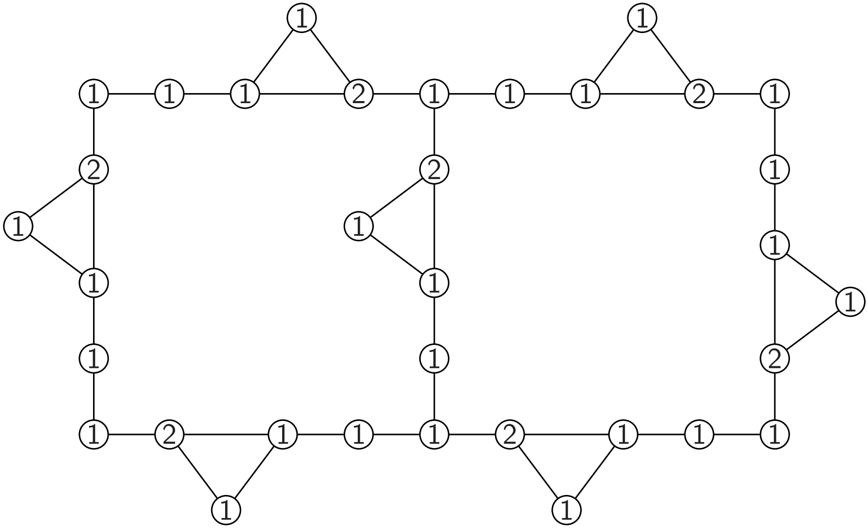

Complexity of Vertex NimG

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- On the RM version, the problem is polynomial.
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Under the misère convention:

- On the MR version, the problem is PSPACE-hard, even when played on planar graphs with maximum degree 4.
- On RM version, the problem is PSPACE-hard, even when played on planar graphs with maximum degree 3.

Perspectives

- What is the complexity of misère GEOGRAPHY on bipartite undirected graphs?
- What is the complexity of VERTEX NIMG-MR on bipartite weighted graphs?
- What about replacing NIM with another game?

Questions?

Thank you.