Misère Geography and Vertex NimG are PSPACE-hard

Gabriel Renault

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Impartial combinatorial games

An impartial combinatorial game is a game:

- with two players
- with complete information
- where there is no chance
- finite (acyclic)
- where both players always have the same sets of moves
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- where there is no chance
- finite (acyclic)
- where both players always have the same sets of moves

We consider two winning conventions:

- in normal version, the player who plays the last move wins
- in misère version, the player who plays the last move loses
GEOGRAPHY
A directed graph $G$ with a token on a vertex.

A move is to slide the token to an out-neighbour and delete the previous vertex.

The game can also be played on an undirected graph.
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Complexity of Geography

Under the normal convention:

- On directed graphs, the problem is $\text{PSPACE}$-complete, even when played on planar bipartite graphs with maximum degree 3. (Lichtenstein, Sipser, 1979)
- On undirected graphs, the problem is polynomial. (Fraenkel, Scheinerman, Ullman, 1993)
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Under the misère convention?
Directed case

Misère Geography and Vertex NimG are PSPACE-hard
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The gadgets

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An example

![Graph Example]

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An example
Losing moves are losing

We need to prove that:

- A move from $u$ to $u'$ is losing.
- A move from $v$ to $uv_7$ is losing.
- If $v$ has been played, the move from $u$ to $uv_1$ is losing.
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Under the misère convention:

- On directed graphs, the problem is \textsc{PSPACE}-complete, even when played on planar bipartite graphs with maximum degree 4.
- On undirected graphs, the problem is \textsc{PSPACE}-complete, even when played on planar graphs with maximum degree 5.
Vertex NimG
A weighted graph $G$ with a token on a vertex.

A move is to slide the token to an out-neighbour and remove some weight from this out-neighbour.

All weights must remain non-negative integers.
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Vertex NimG-MR

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All weights must remain non-negative integers.
A weighted graph $G$ with a token on a vertex.

A move is to remove some weight from the current vertex and slide the token to an out-neighbor.

All weights must remain non-negative integers.
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Complexity of Vertex NimG

Under the normal convention:

- On the MR version, the problem is \textsc{pSpace}-hard, even when played on planar graphs with maximum degree 3. (Burke, George, 2011)
- On the RM version, the problem is polynomial. (Duchêne, R., 2014)
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Misère Geography and Vertex NimG are PSPACE-hard
RM Gadgets

Misère Geography and Vertex NimG are \textsc{PSPACE}-hard.
Example

Misère Geography and Vertex NimG are $\text{PSPACE}$-hard.
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- On RM version, the problem is \texttt{PSPACE}-hard, even when played on planar graphs with maximum degree 3.
What is the complexity of misère Geography on bipartite undirected graphs?

What is the complexity of Vertex NimG-MR on bipartite weighted graphs?

What about replacing Nim with another game?
Questions?

Thank you.