Coloring perfect graphs with bounded clique number

Joint work with M. Chudnovsky, P. Seymour and S. Spirkl

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LIP, ENS Lyon

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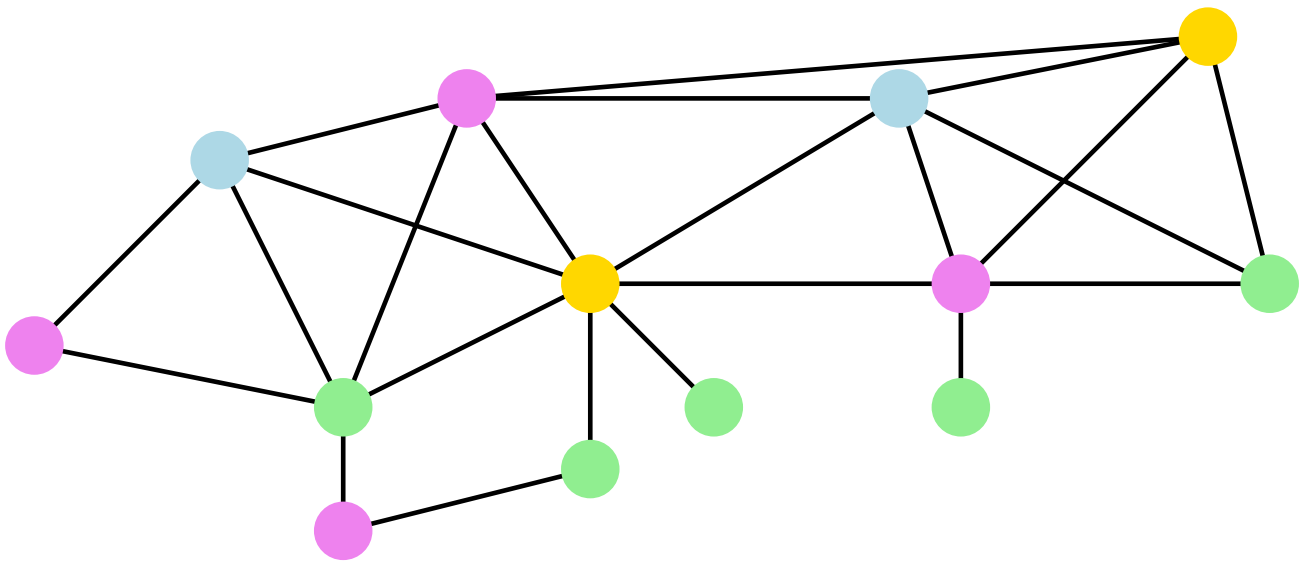
Séminaire AIGCo, LIRMM

A **proper** k**-coloring** of G is an assignment of colors from $\{1, \ldots, k\}$ such that any two adjacent vertices are given different colors.



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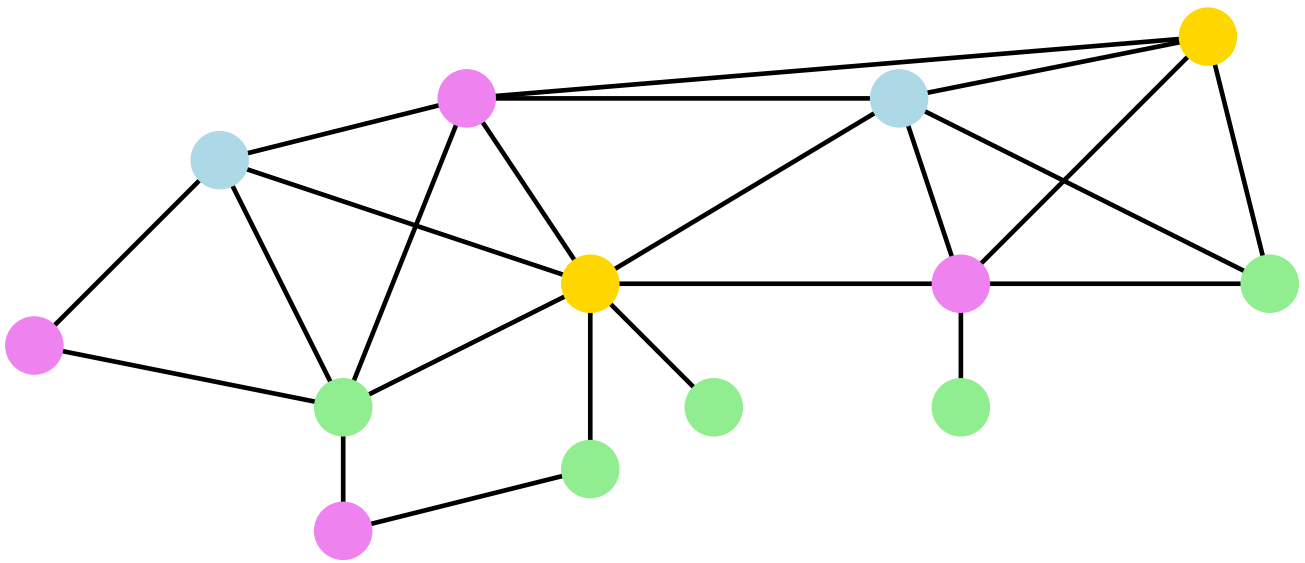

Graph Coloring

Input: A graph G and an integer k.
Output: Does G admits a proper k-coloring?

Graph Coloring is NP-complete.

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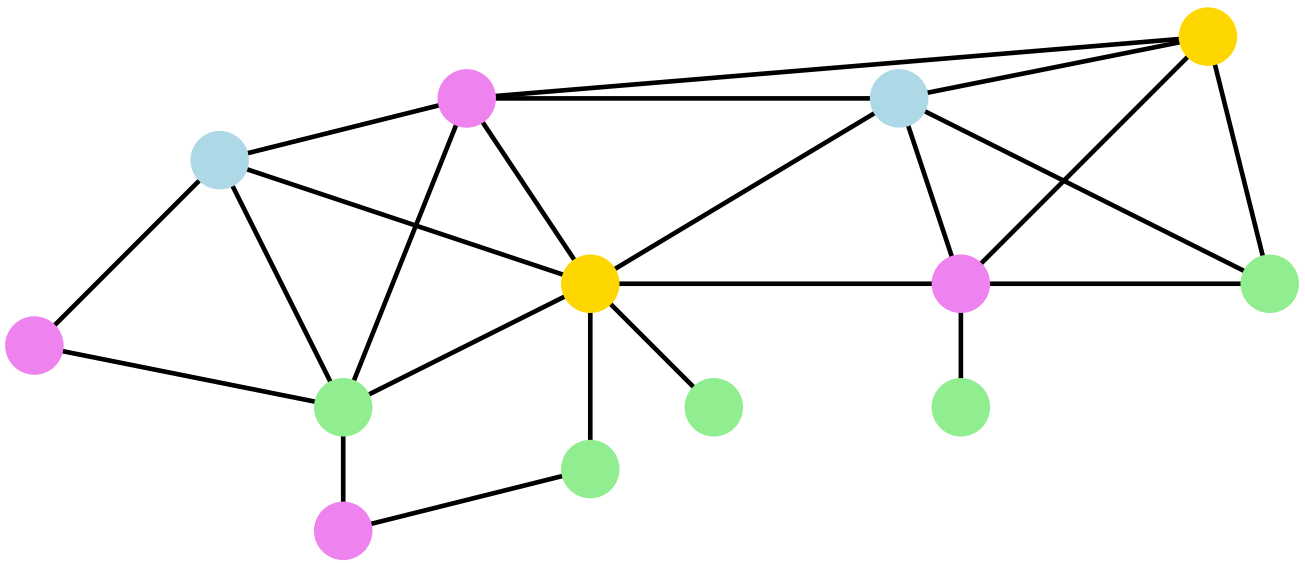

~~Graph Coloring~~ 3-Coloring

Input: A graph G ~~and an integer k.~~

Output: Does G admits a proper ~~k-coloring?~~ 3-coloring?

Graph Coloring is NP-complete. Even 3-Coloring is!

- $\chi(G)$: **chromatic number** of G, i.e. minimum number of color in a proper coloring.
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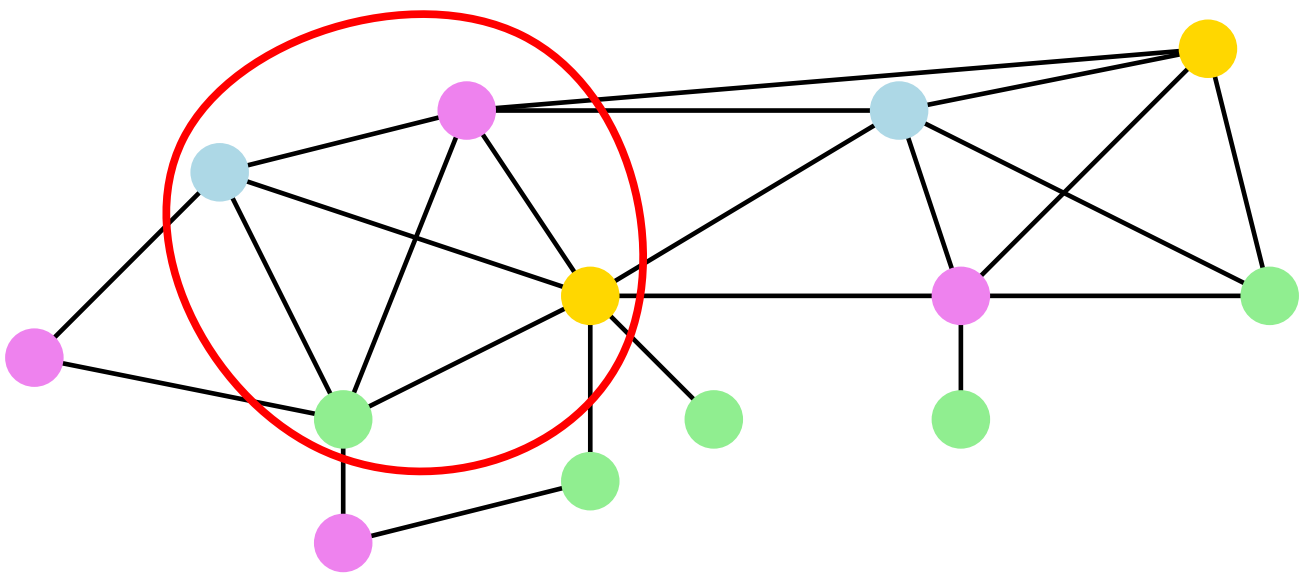
$$\chi(G) \geq \omega(G)$$



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G is **perfect** if and only if $\chi(G) = \omega(G)$ and the equality holds for every induced subgraph H of G.

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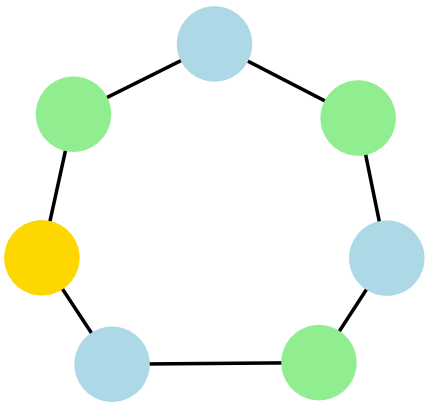
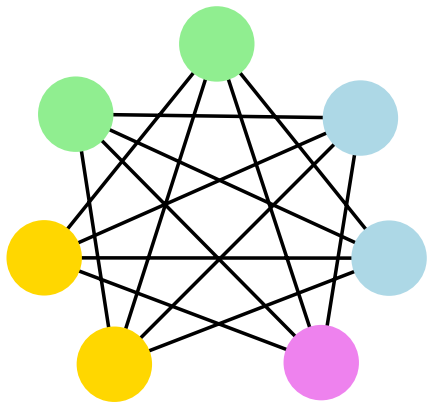


Odd hole C_7



Odd antihole $\overline{C_7}$

Berge's Conjecture (1960's)

A graph G is perfect if and only if G contains no odd hole and no odd antihole as induced subgraph ($=G$ is *Berge*).

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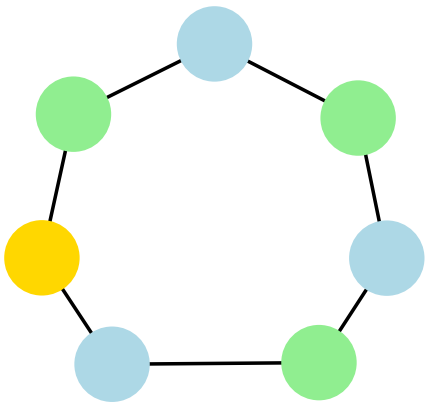

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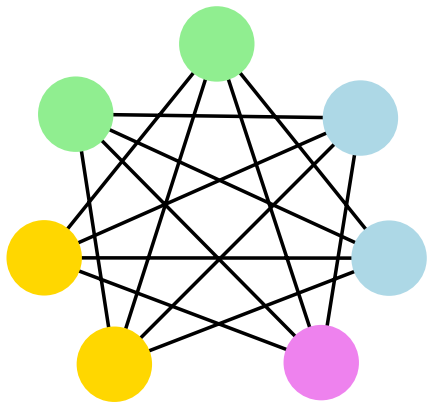

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Berge's Conjecture (1960's) $\Rightarrow$ Strong Perfect Graph Theorem

A graph G is perfect if and only if G contains no odd hole and no odd antihole as induced subgraph ($=G$ is *Berge*).

Proved in 2002 by Chudnovsky, Robertson, Seymour and Thomas.

What about coloring perfect graphs?

Theorem [Grötschel, Lovász, Schrijver 1981]

The Maximum Weighted Stable Set problem can be solved in polynomial time for perfect graphs.

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Not satisfying! We know so much on perfect graphs that we want a *combinatorial* algorithm.

We know *so much*?

Decomposition theorem from [CRST'02]

If G is Berge, then

- either G is *basic* (bipartite, line graph of bipartite, ...),
- or G can be *decomposed* in a given way.

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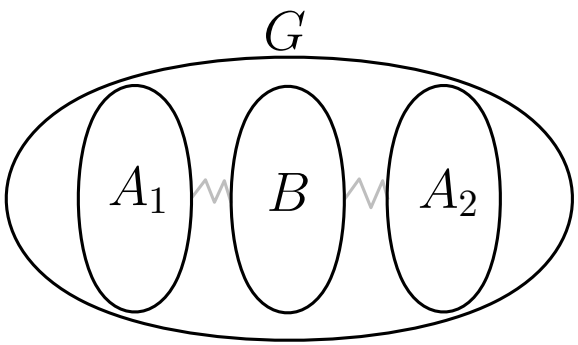

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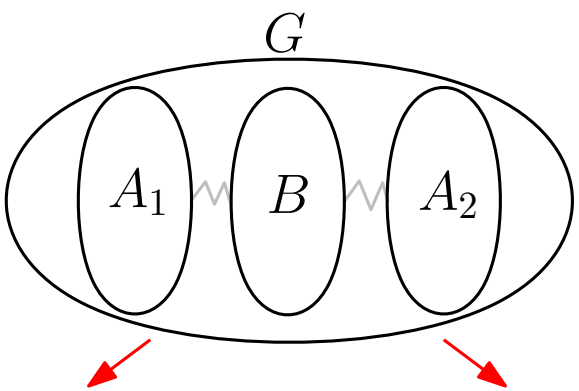

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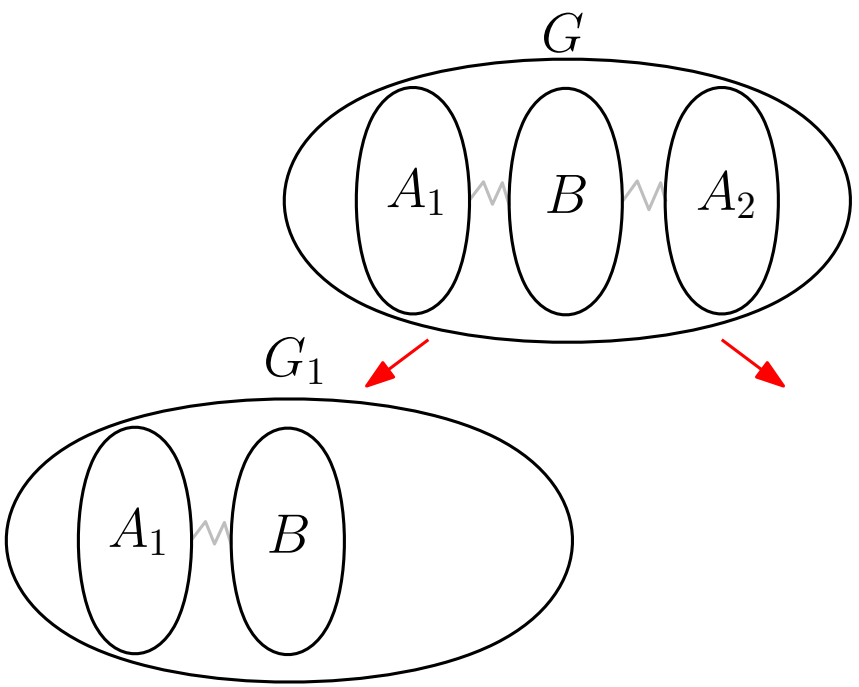

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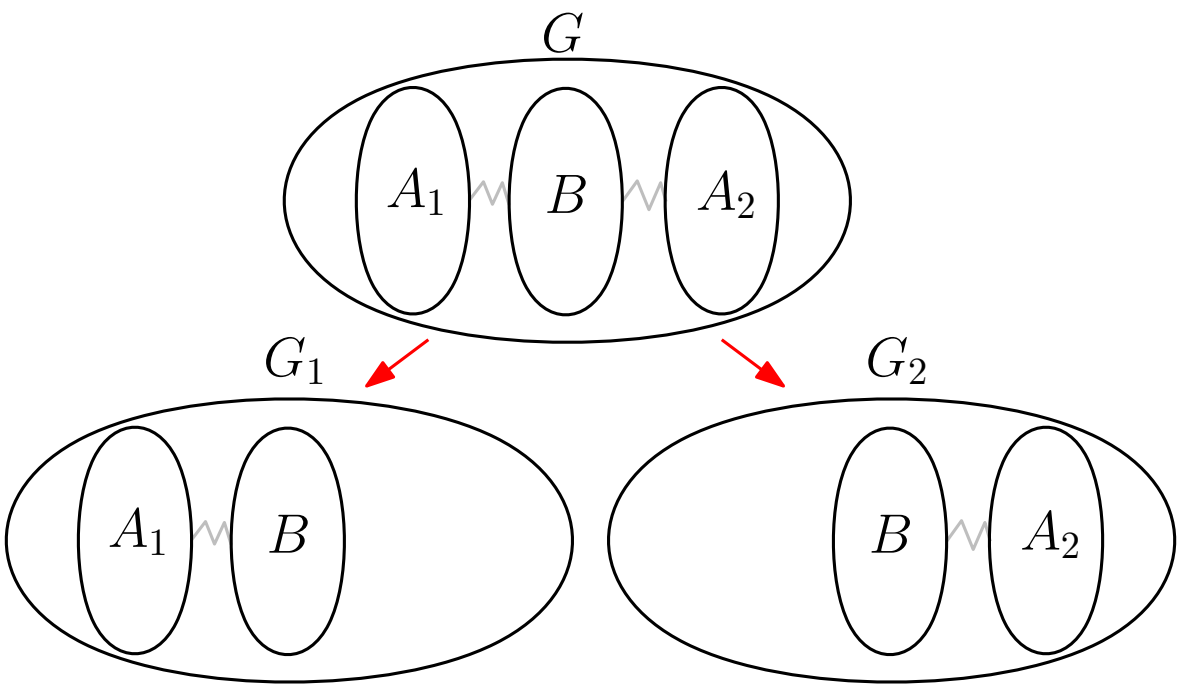

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$\Rightarrow$ Contradiction!

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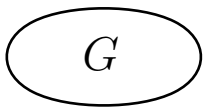

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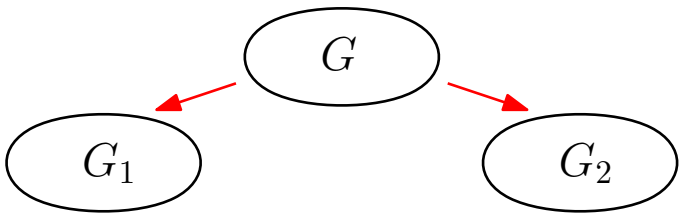

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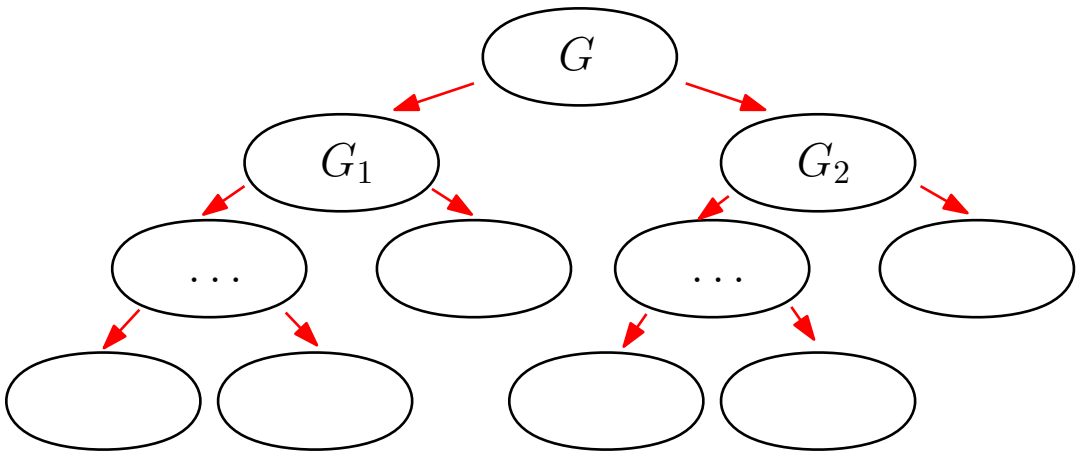

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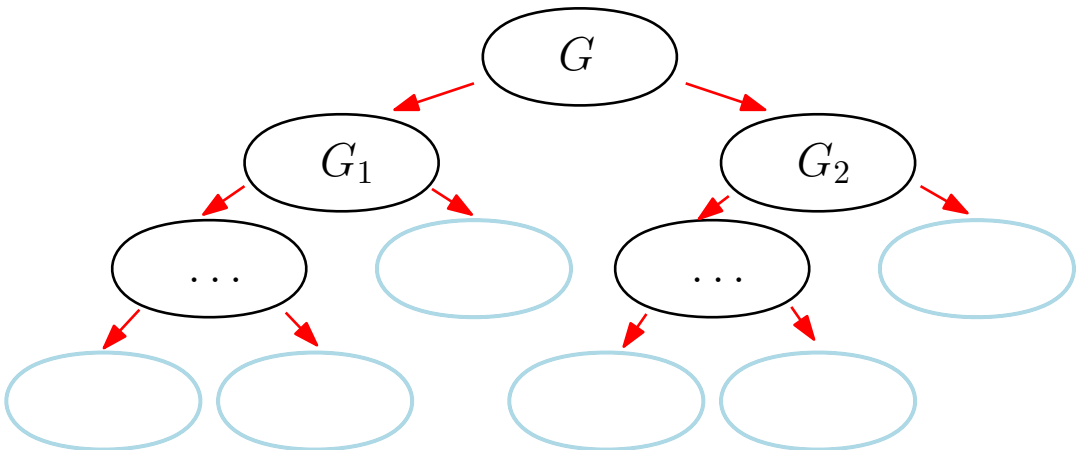

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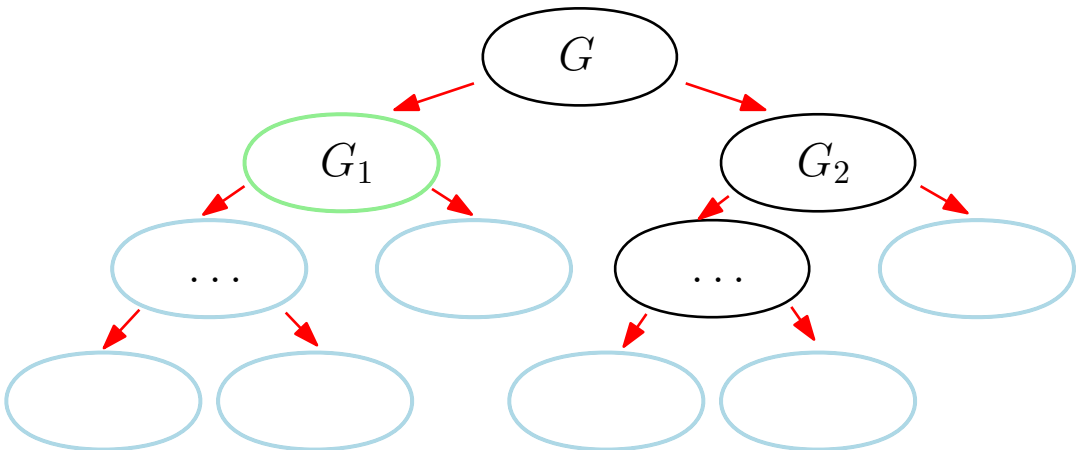

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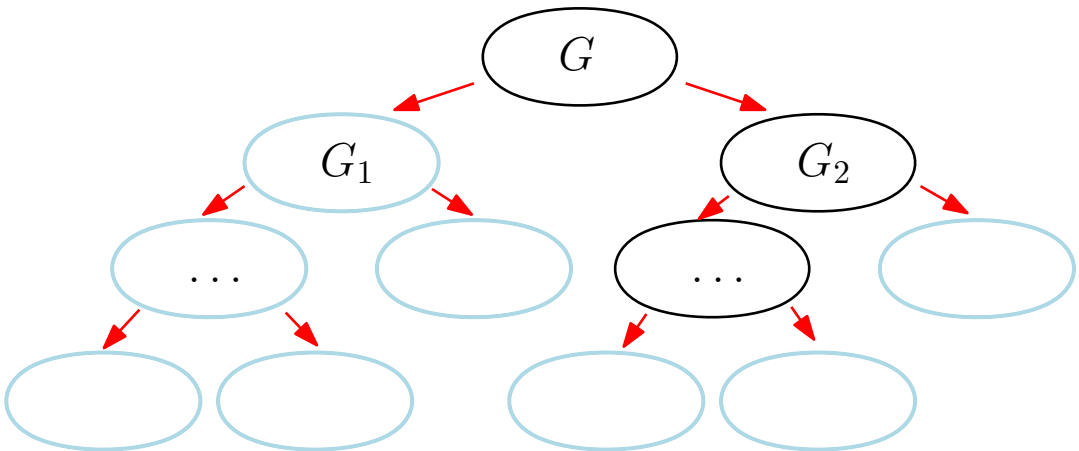

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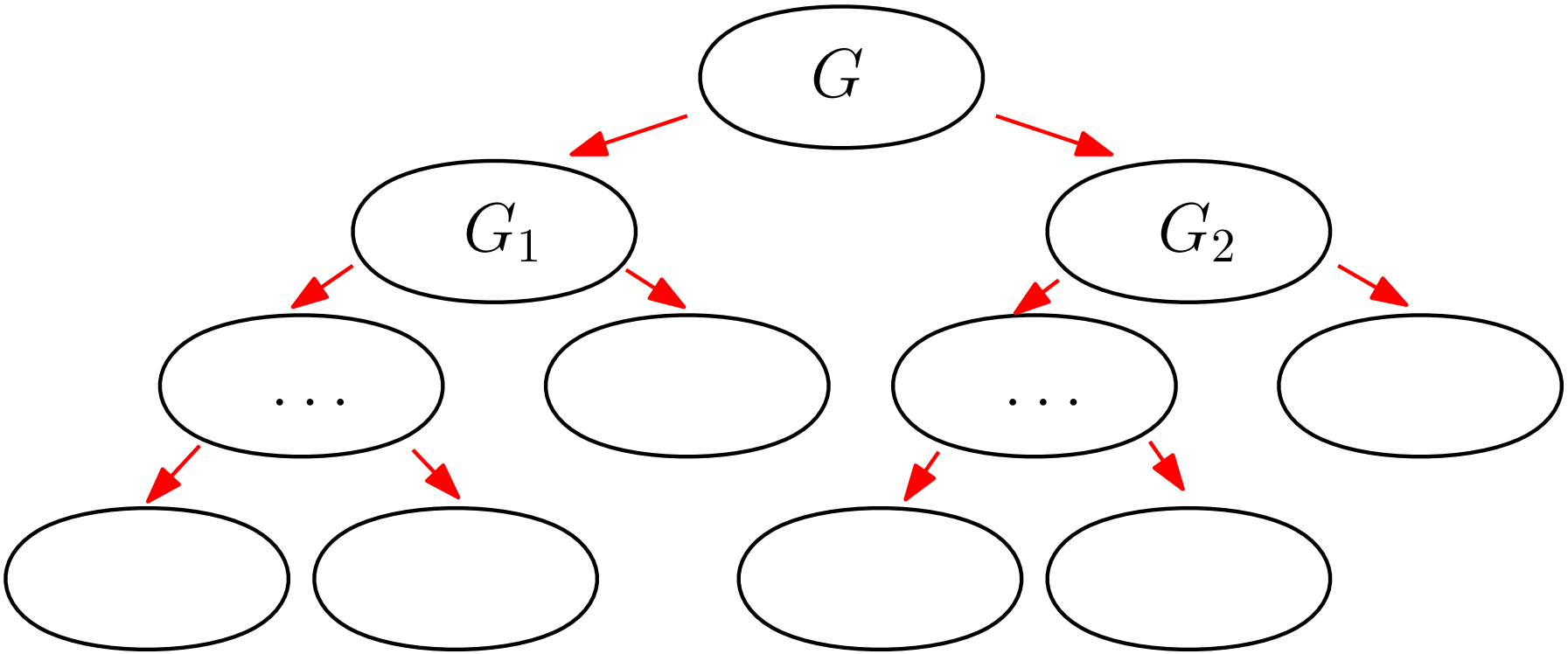
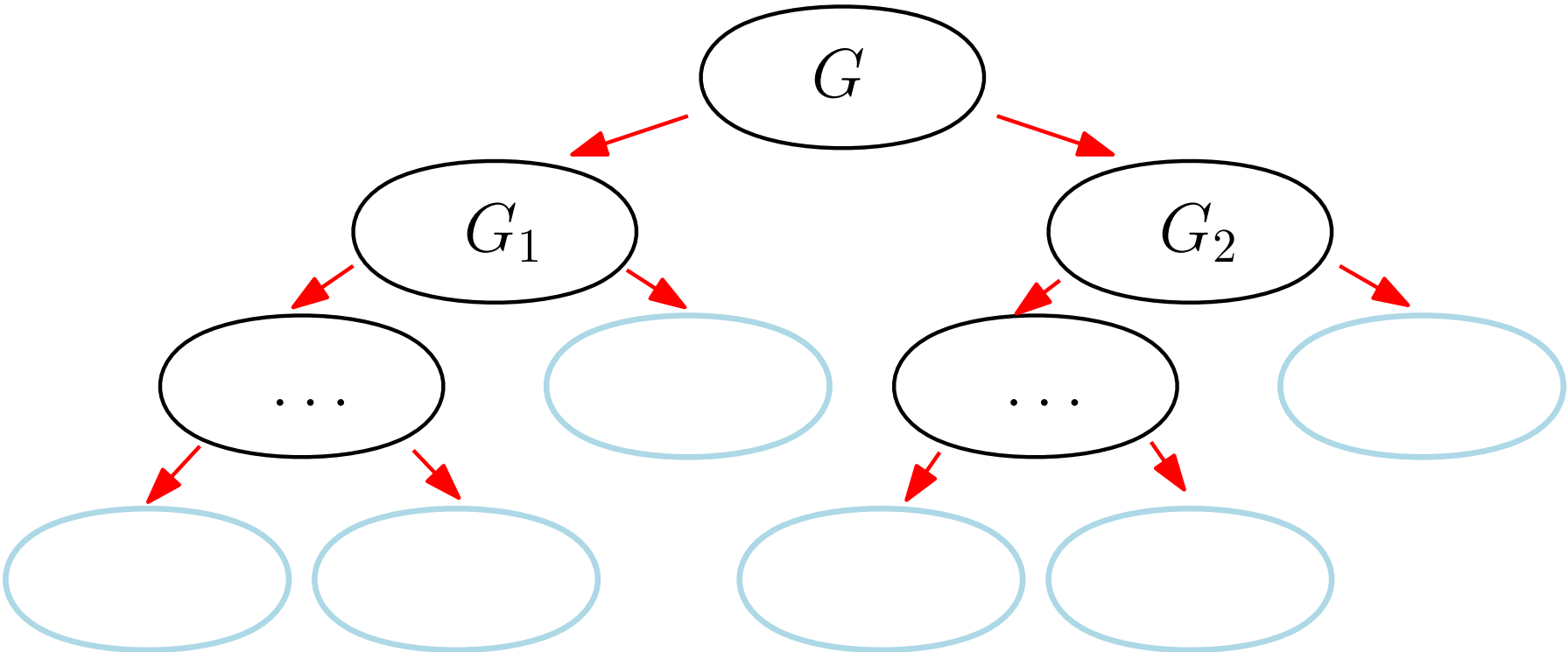
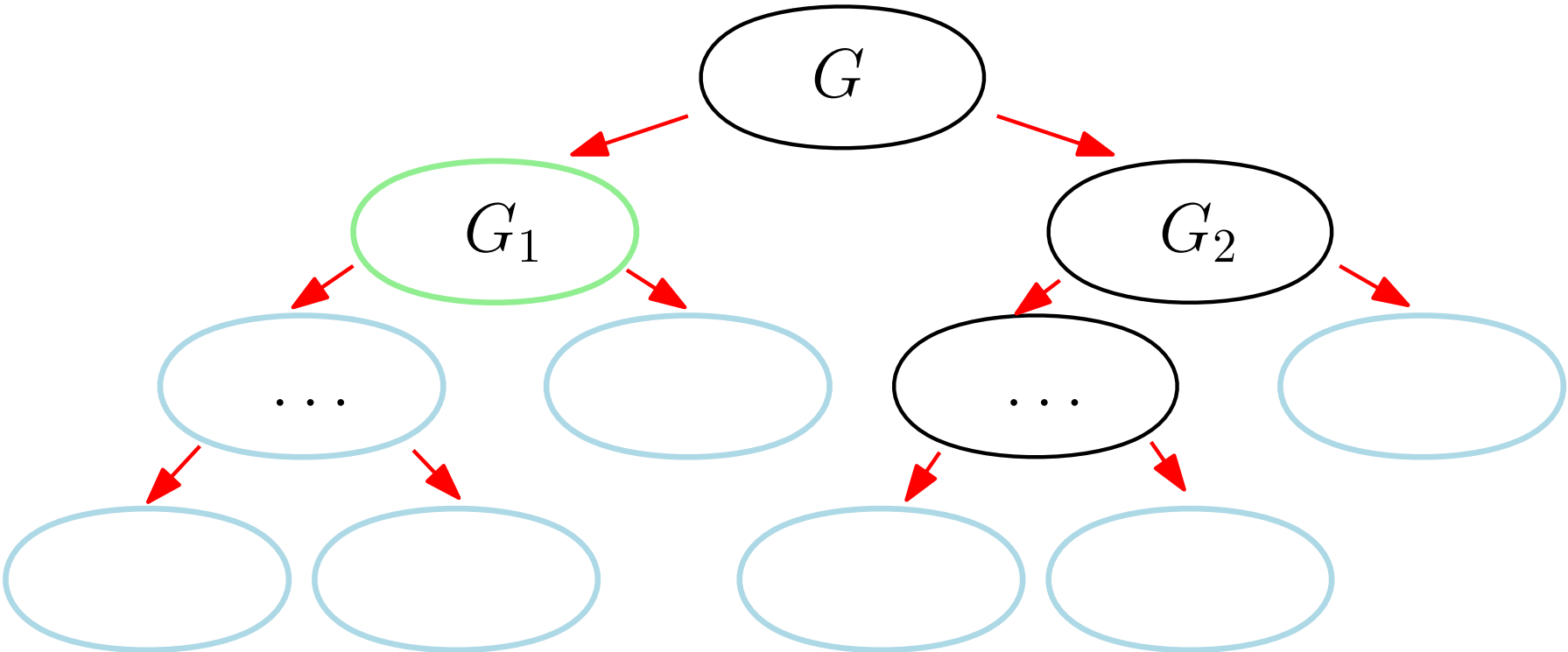


Hence four intermediate steps to reach:



Hence four intermediate steps to reach:

- Know how to directly solve the problem on leaves.



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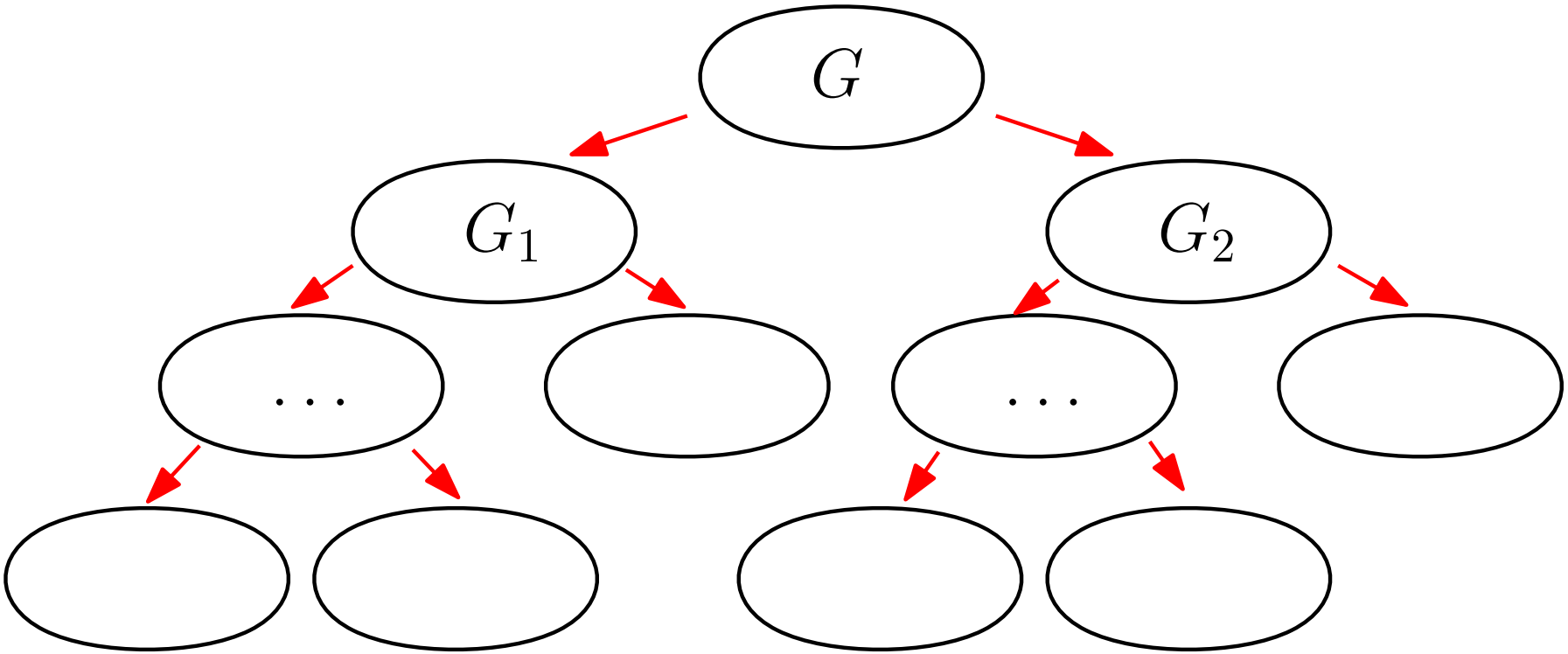

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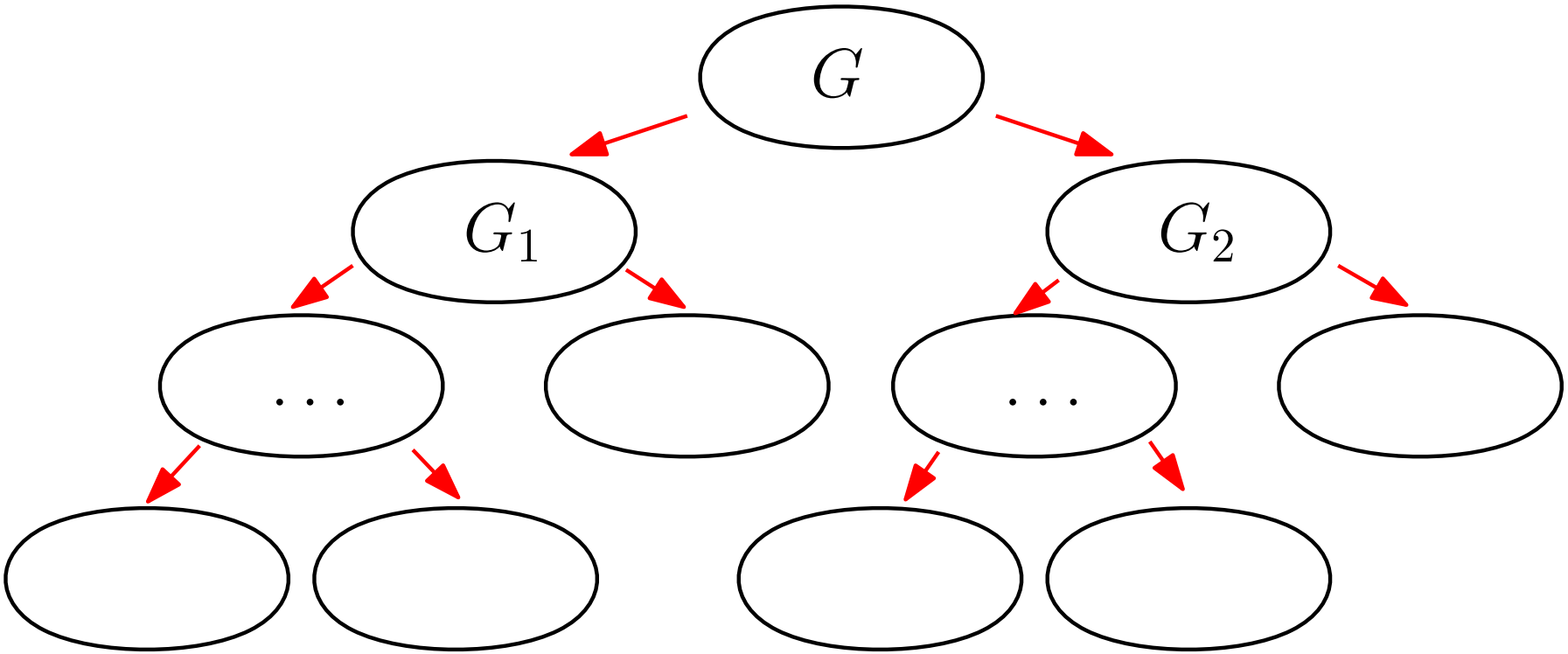
- Bound the size of the tree by a polynomial in n.



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- Know how to algorithmically construct the tree.

Our result

Theorem [Chudnovsky, L., Seymour, Spirkl]

We design an algorithm with the following specification:
Algorithm $\mathcal{A}_k$:
Input: A perfect graph G with $\omega(G) \leq k$.
Output: A proper coloring of G with $\chi(G) = \omega(G)$ colors.
Running time: $\mathcal{O}(n^{(\omega(G)+1)^2})$

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We proceed by induction on $k \rightarrow$ we can call $\mathcal{A}_{k-1}$ when needed.

Previous results in this direction:

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A combinatorial algorithm that optimally colors:

- *any Berge graph with no BSP*

  [Chudnovsky, Trotignon, Trunkc, Vušković 2015]

- *any C_4-free Berge graph*

  [Chudnovsky, Lo, Maffray, Trotignon, Vušković 2015$^+$]

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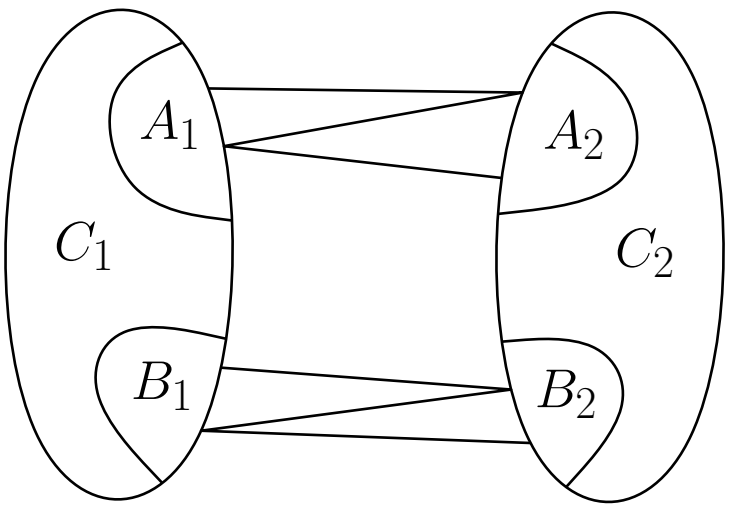

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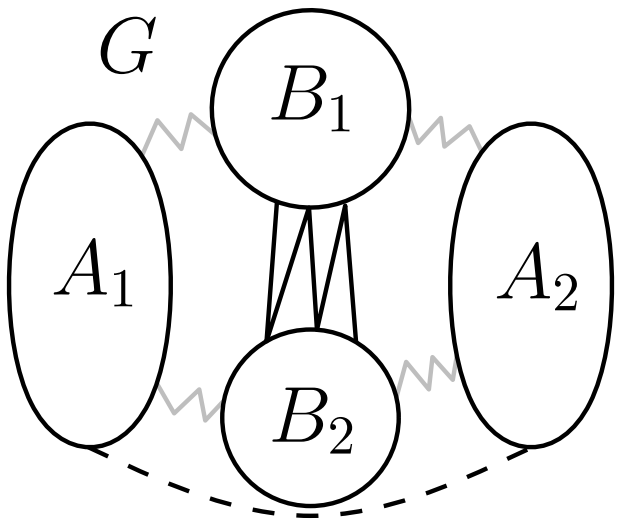
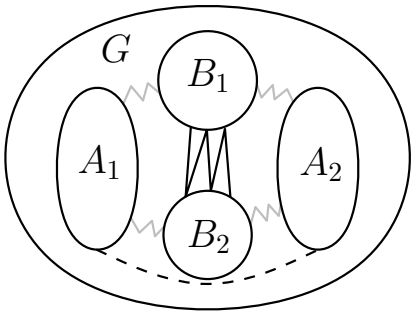
- G or $\overline{G}$ admits a decomposition by 2-join,
- G admits a decomposition by balanced skew partition (BSP).



Skew partition

Our decomposition tree



Decompose along BSP until the graph:

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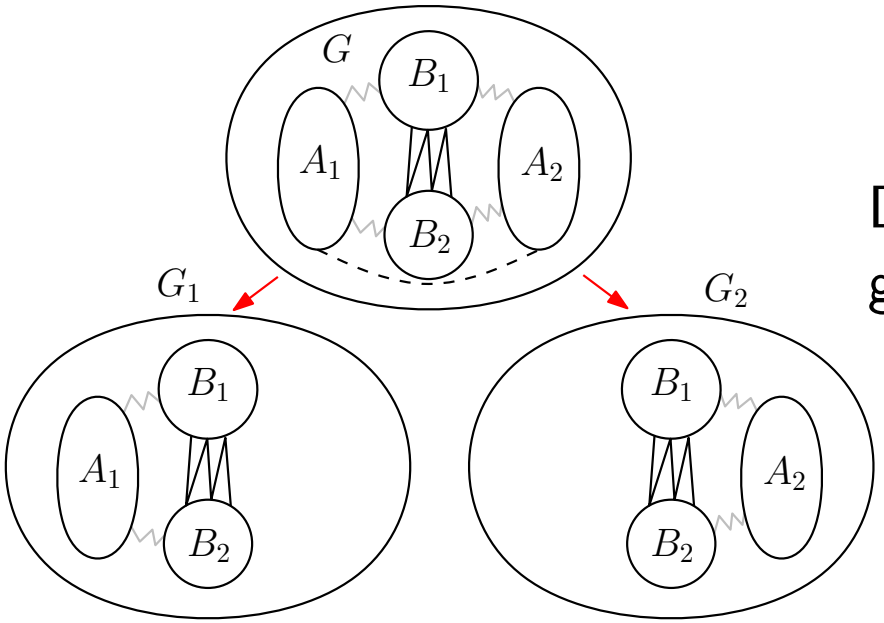

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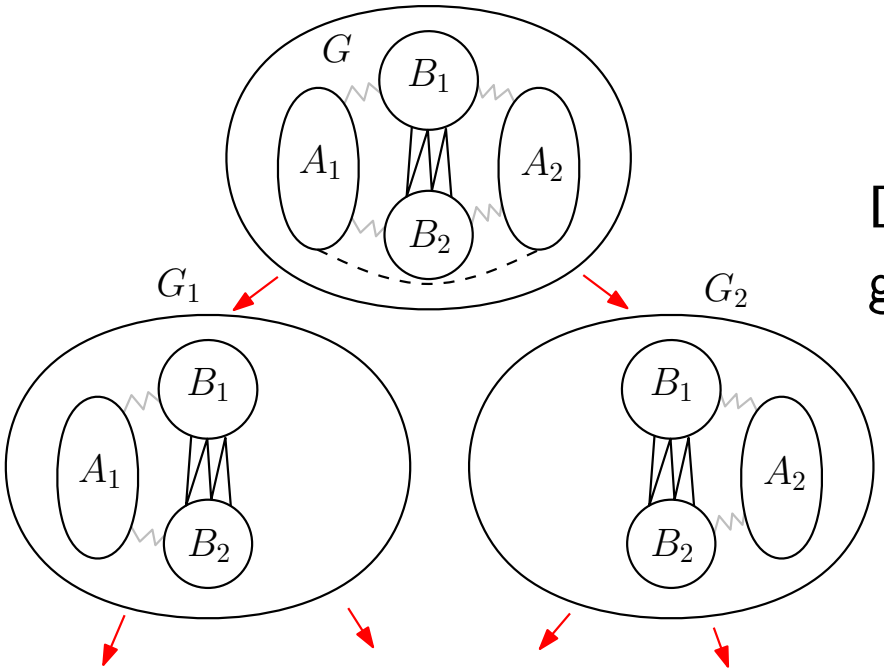

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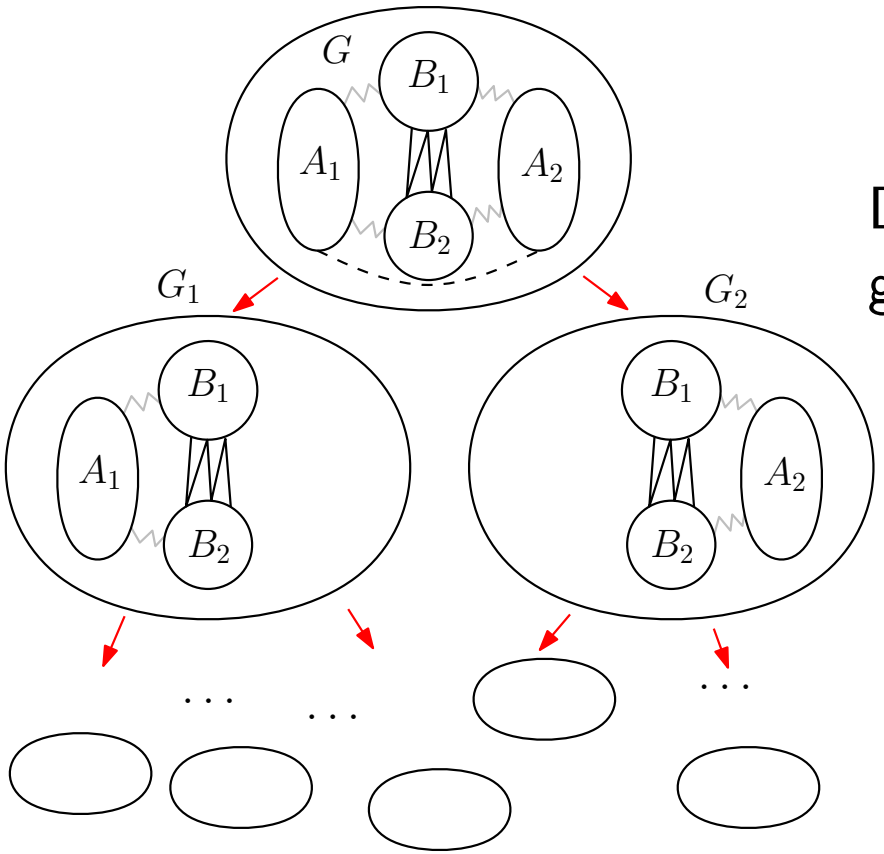

Decompose along BSP until the graph:

- admits no BSP,
- or is not anticonnected,
- or has clique number $< k$,
- or has bounded size $< 2k$.

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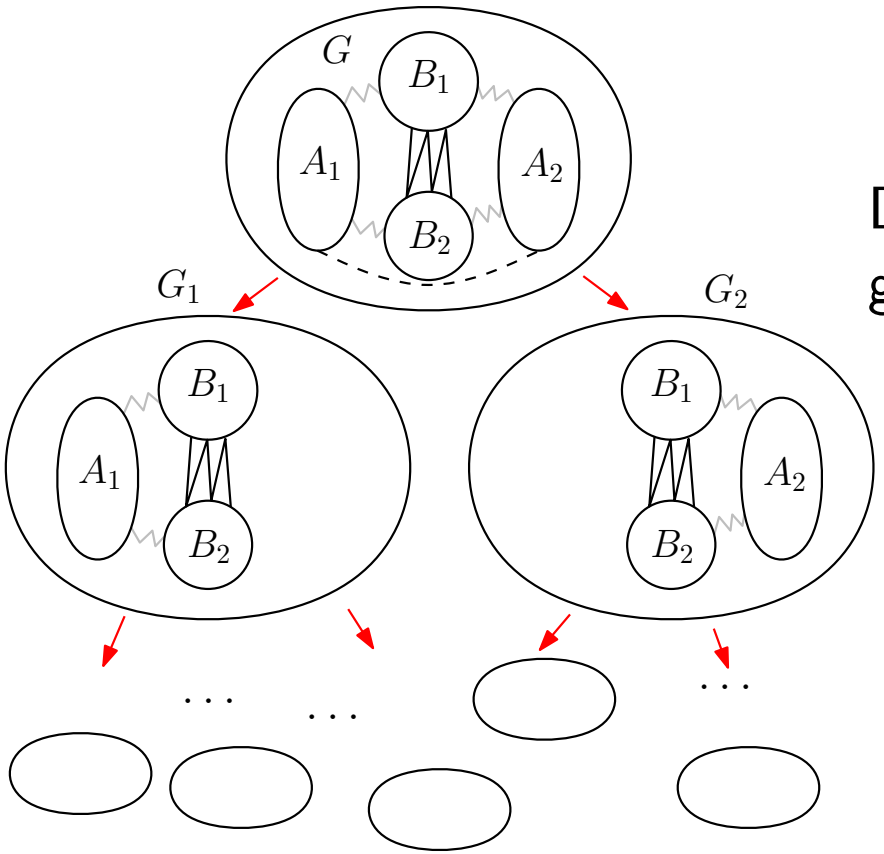

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Color with CTTV algo $\rightarrow \mathcal{O}(n^7)$

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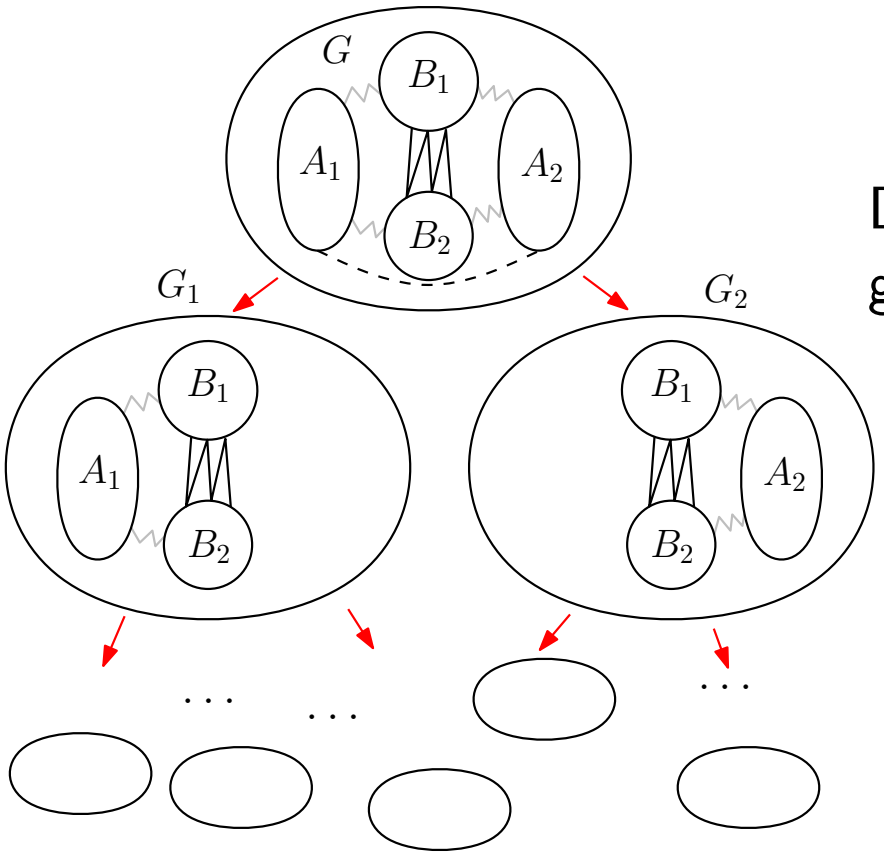

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Color with $\mathcal{A}_{k-1} \rightarrow \mathcal{O}(n^{\omega(G)^2})$

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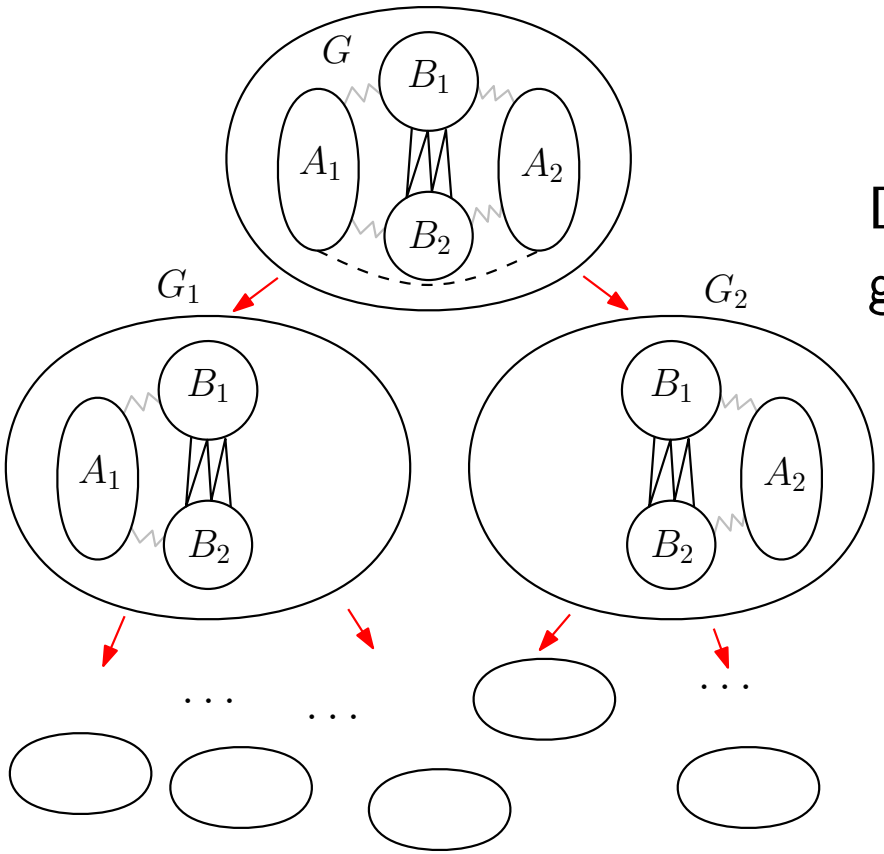

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Color with CTTV algo $\rightarrow \mathcal{O}(n^7)$
Color with $\mathcal{A}_{k-1} \rightarrow \mathcal{O}(n^{\omega(G)^2})$
Easy to color

Outline

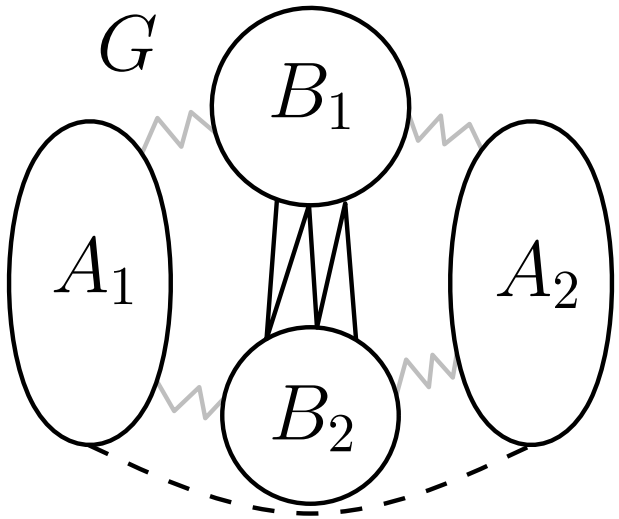
Four intermediate steps to reach:

- ✓ Know how to directly solve the problem on leaves
  $\rightarrow$ time $\mathcal{O}(n^{\max(7,\omega(G)^2)})$
- Know how to go from children to father (combining solutions)

- Bound the size of the tree by a polynomial in n
- Know how to algorithmically construct the tree

How to combine solutions?

Problem when gluing solutions



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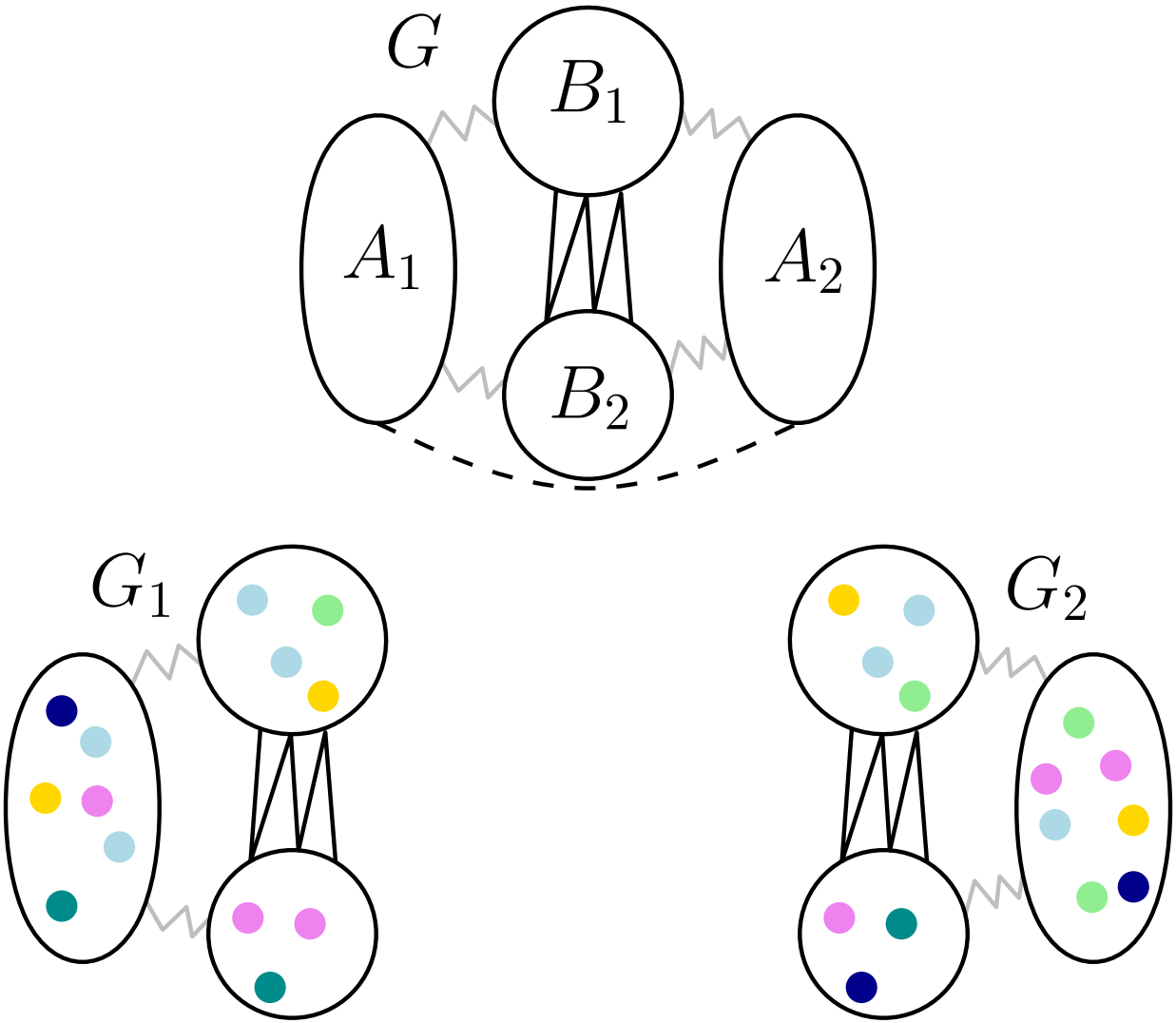

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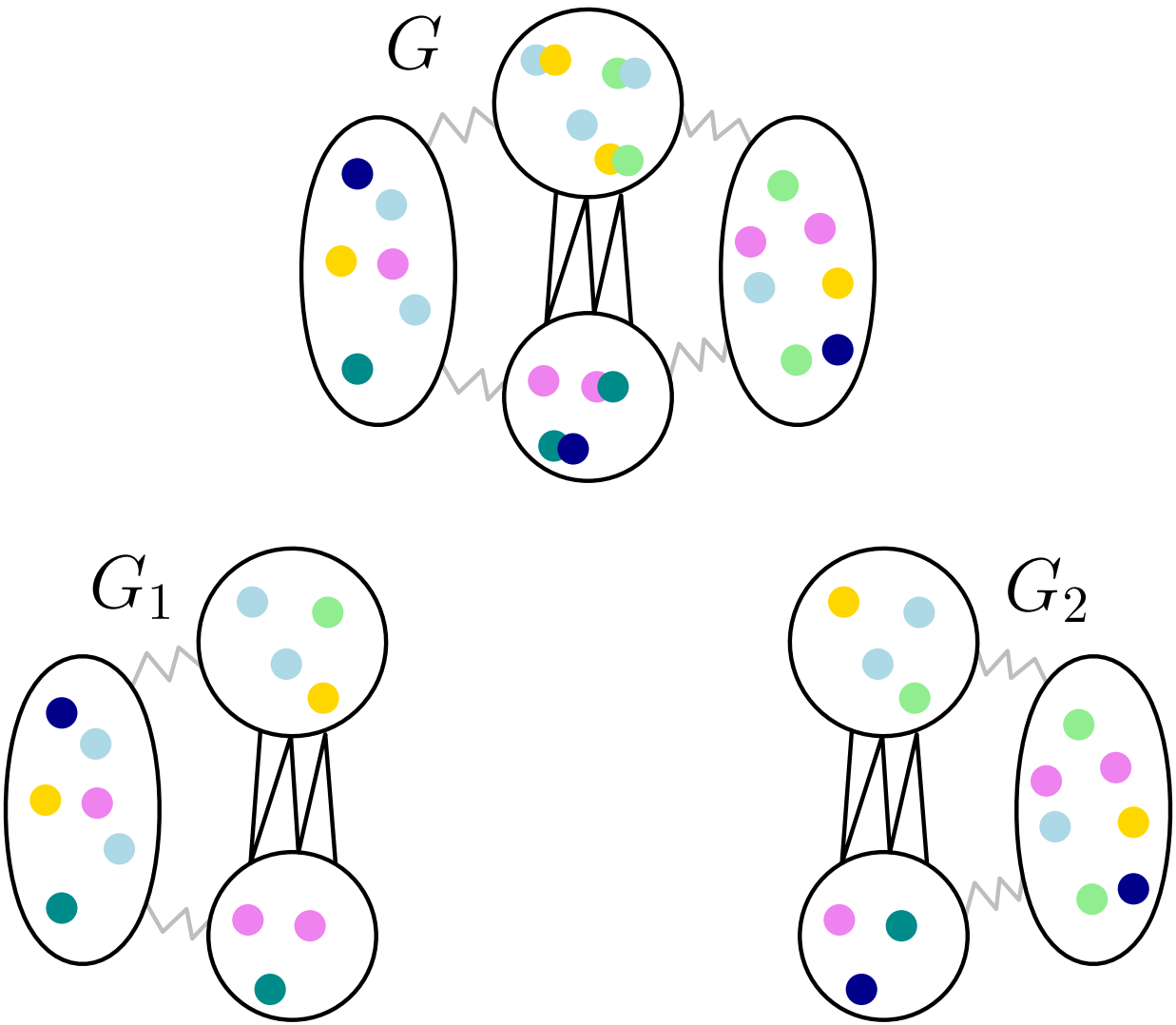

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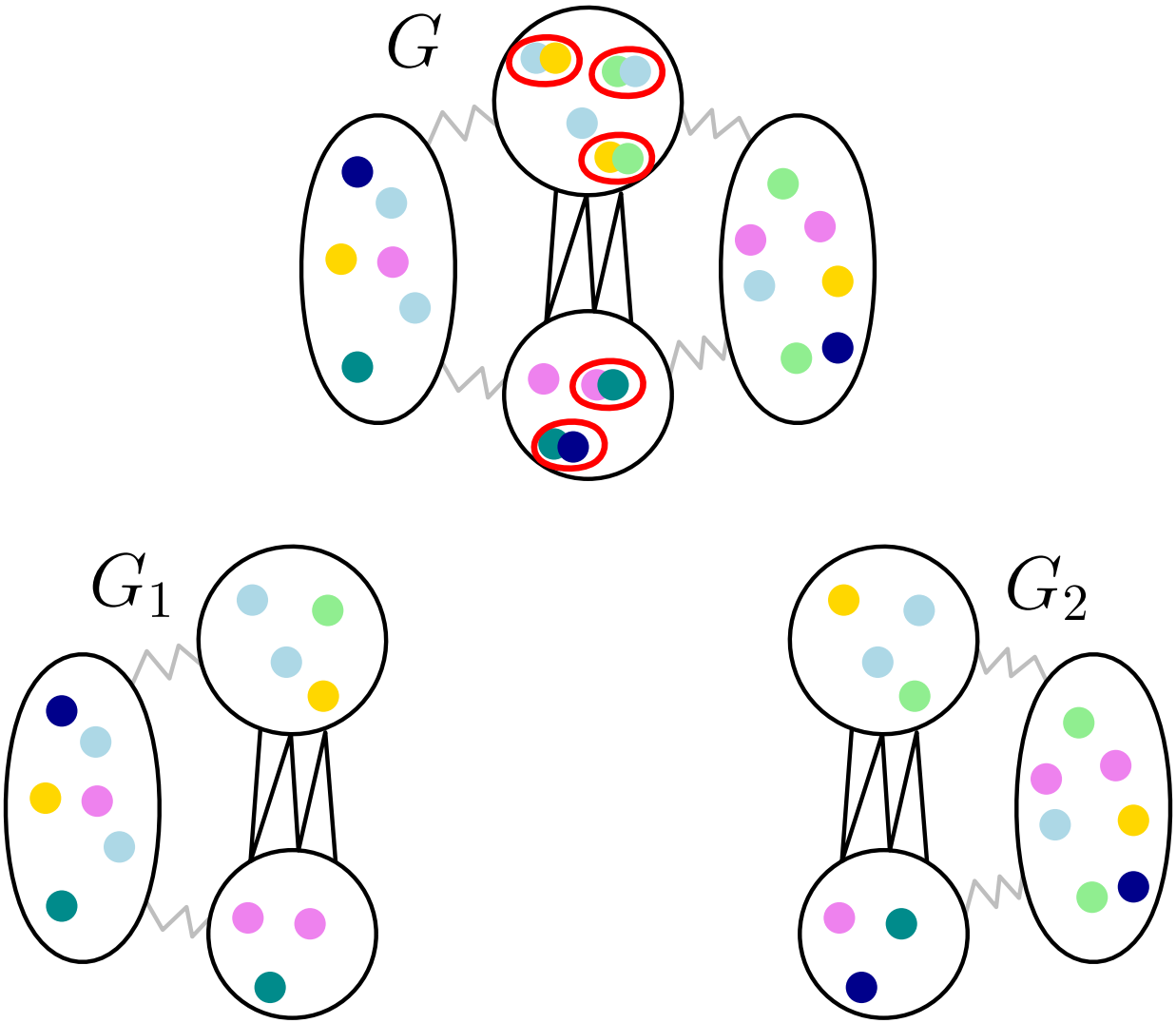

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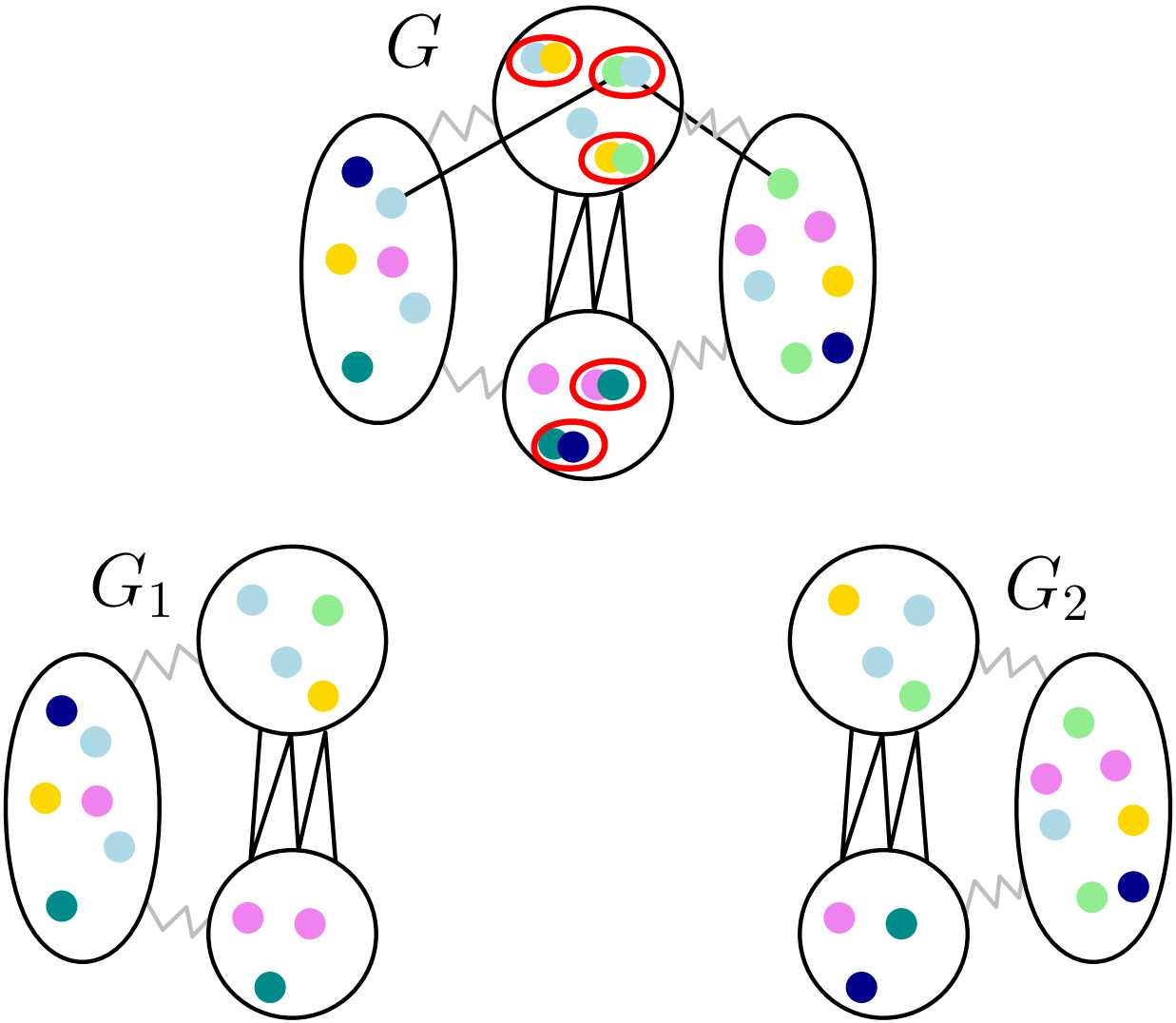

Goal: Find a partition in two sets (F_1, F_2):

- F_1 is k_1-colorable, $k_1 < \omega(G)$;
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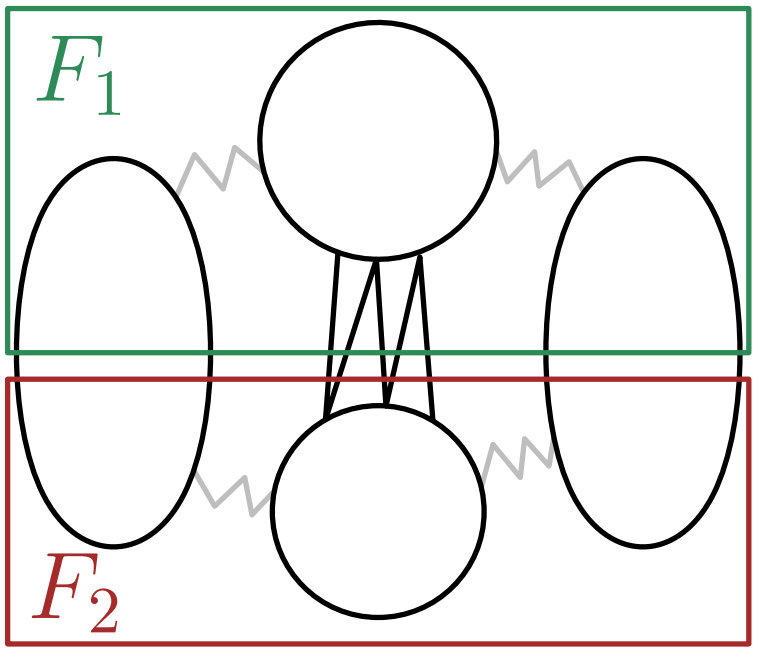

Call $\mathcal{A}_{k-1}$ on F_1 and F_2. This gives a proper $\omega(G)$-coloring of G.

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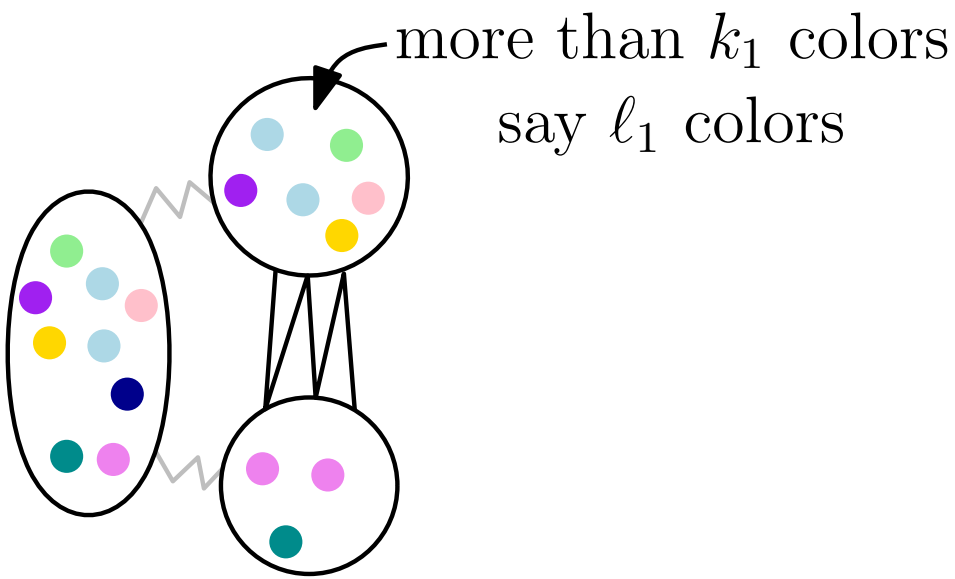

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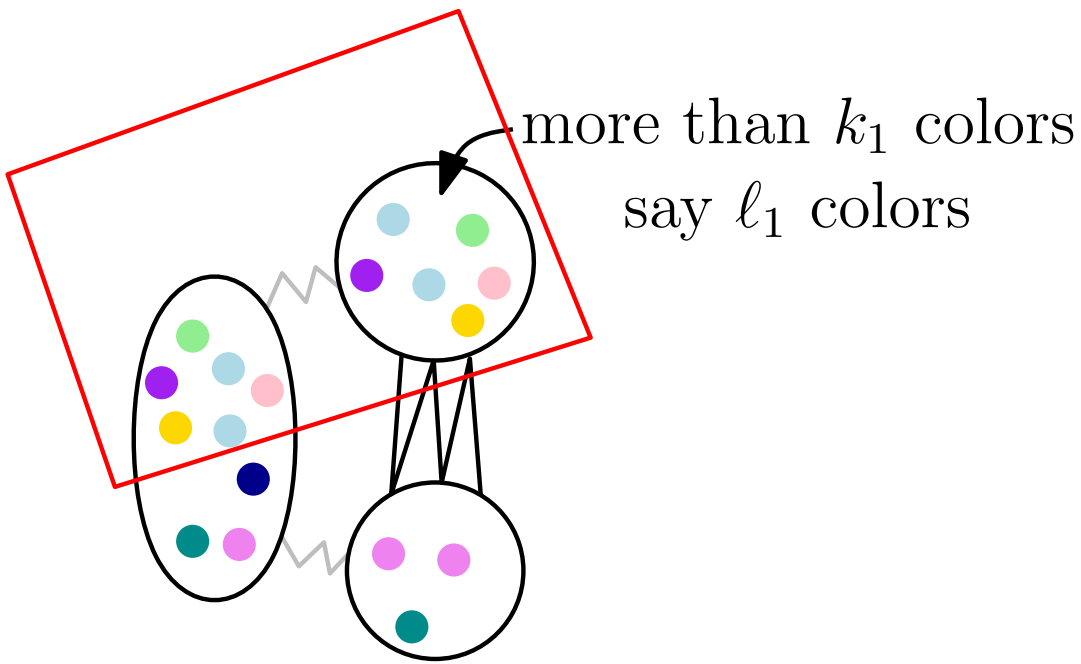

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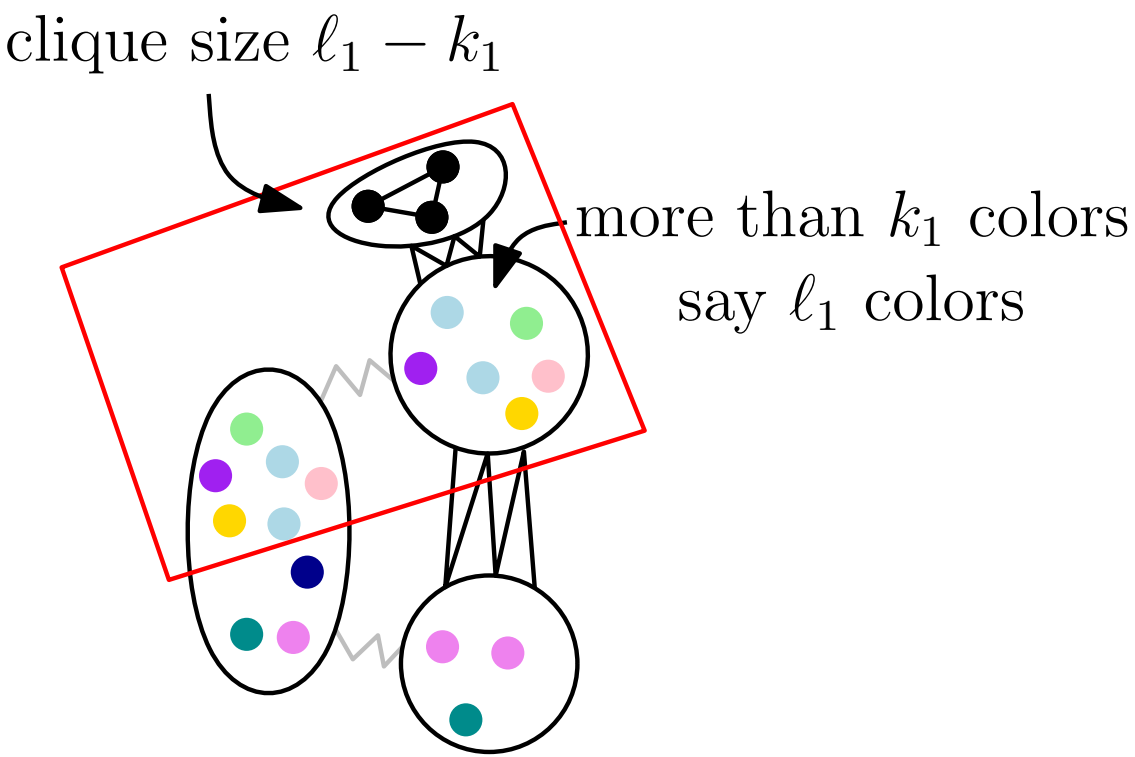

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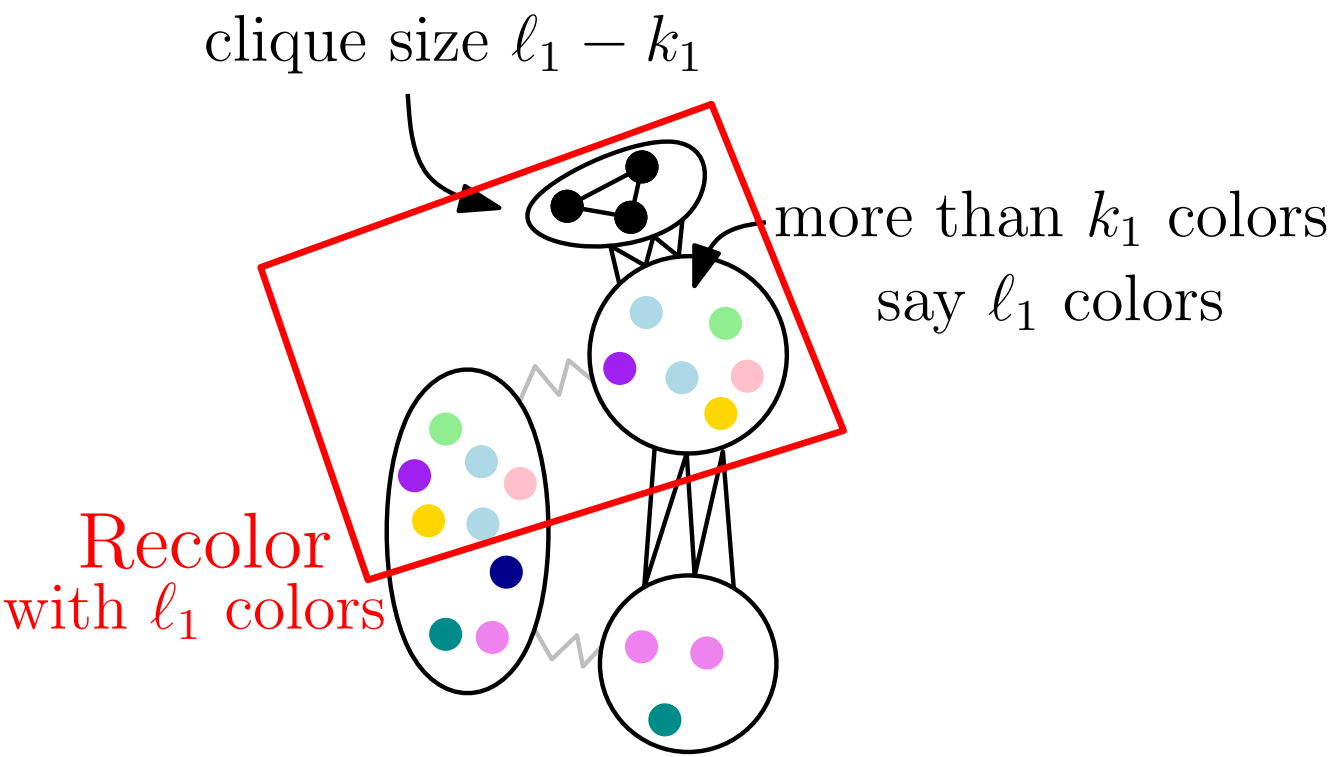

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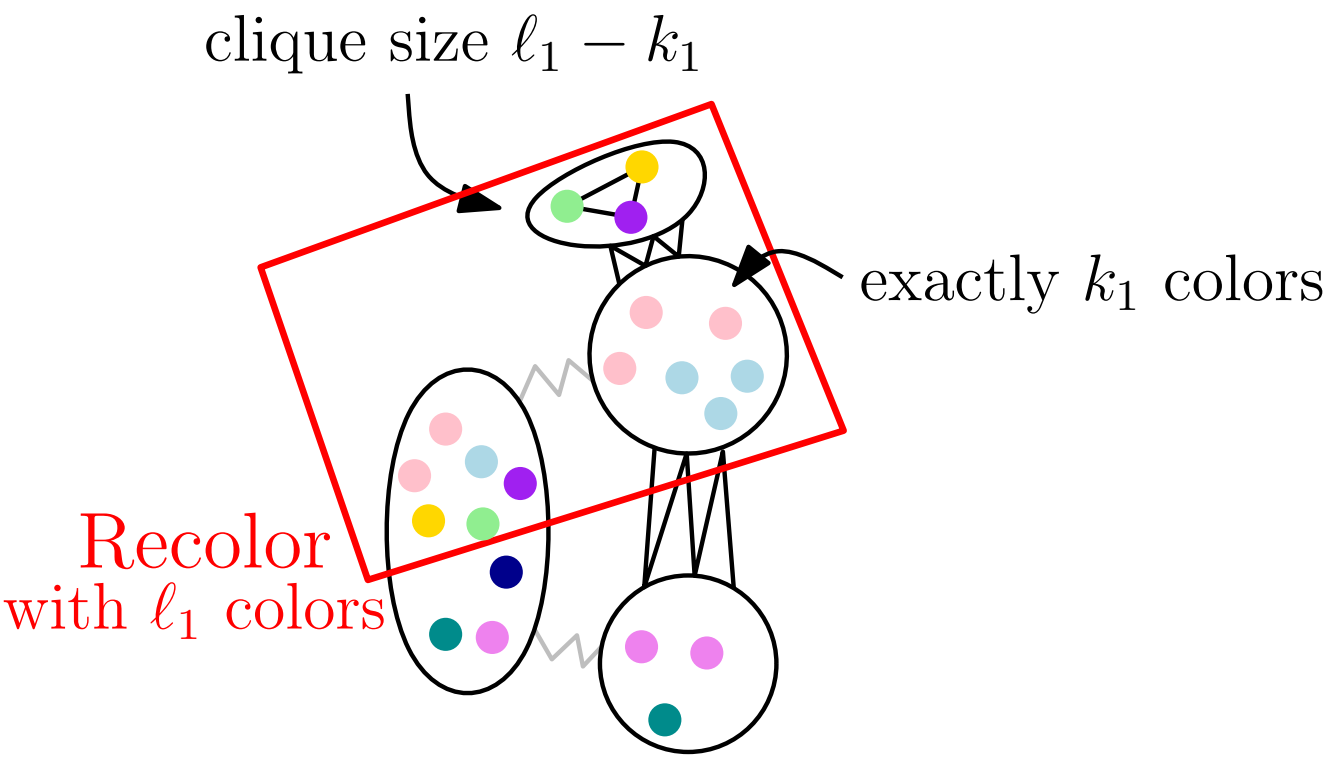

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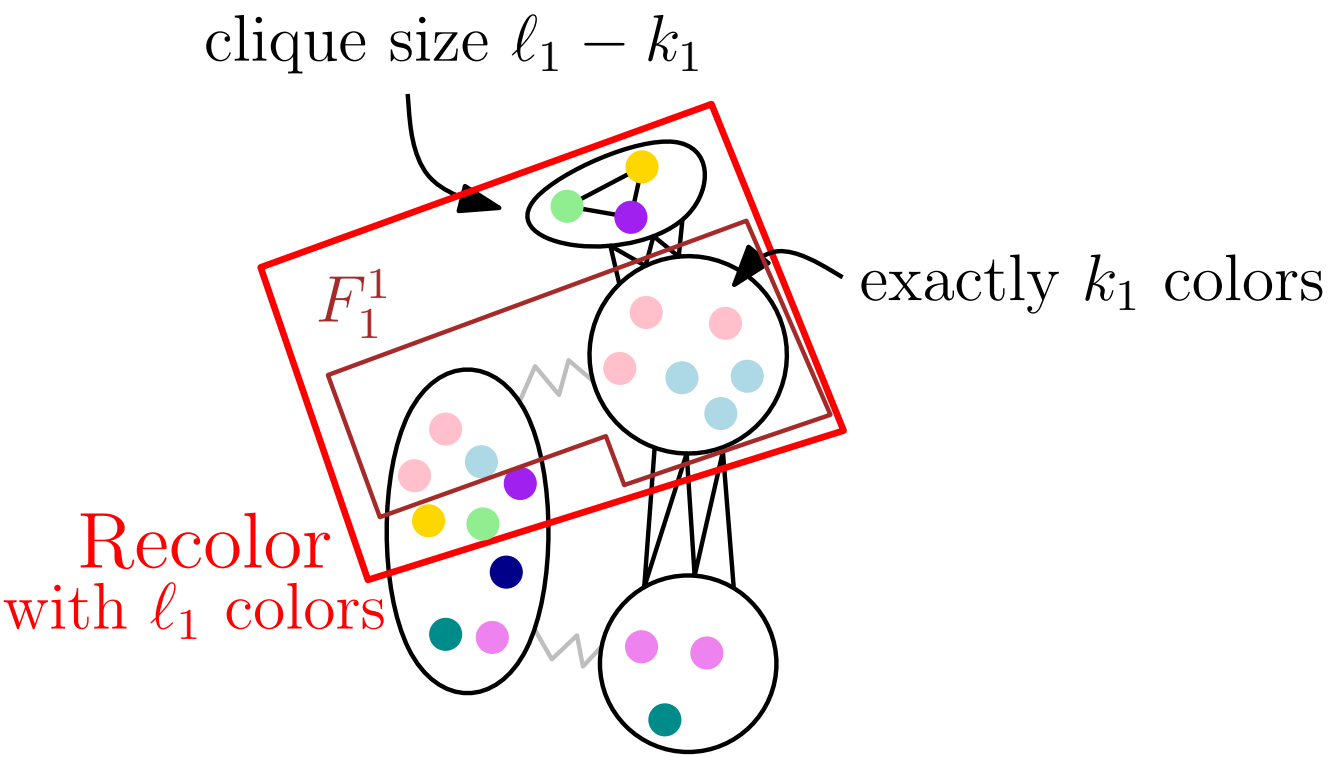

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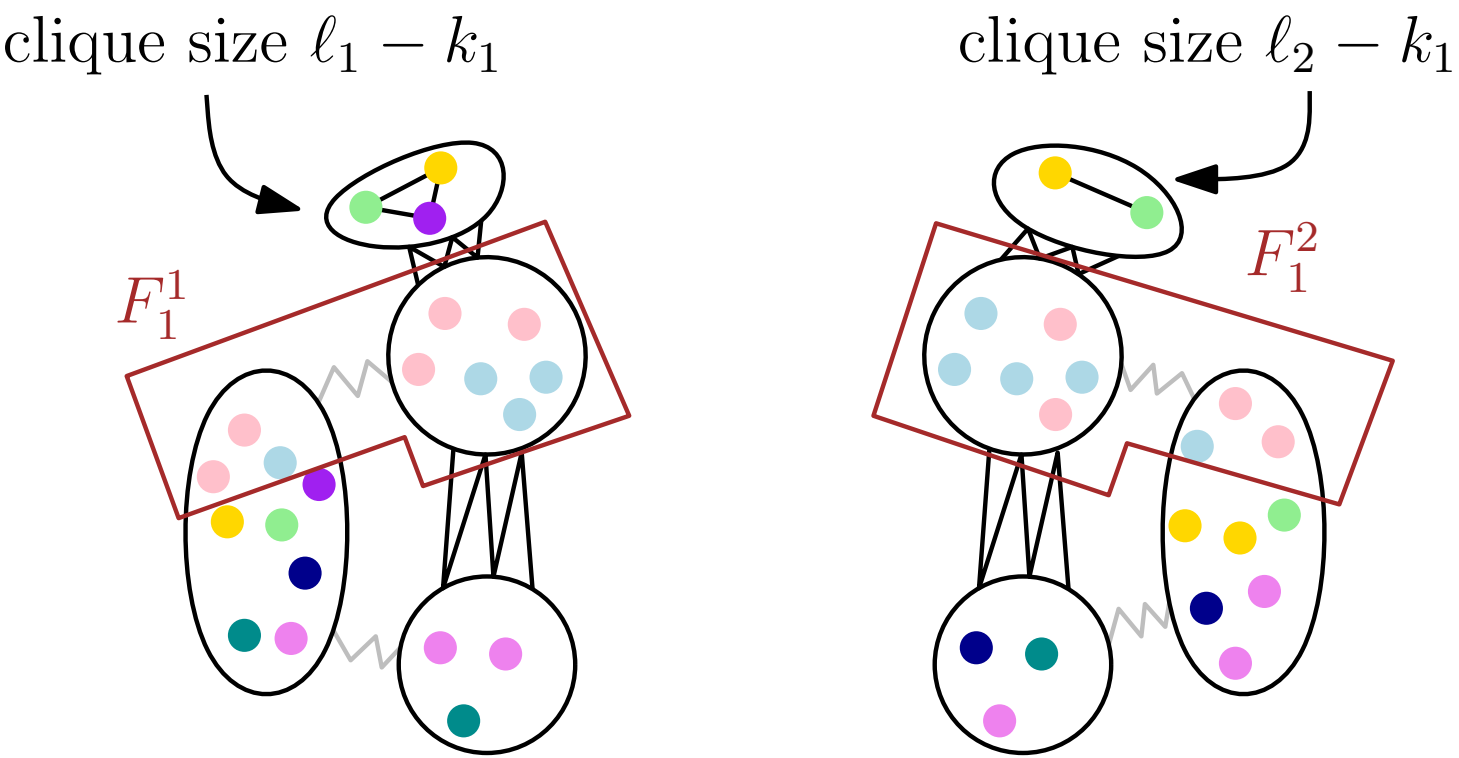

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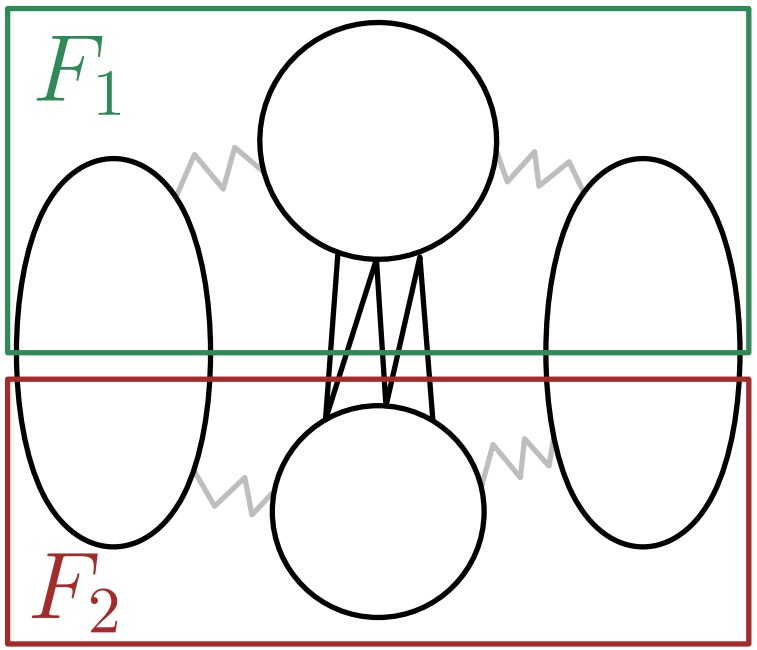

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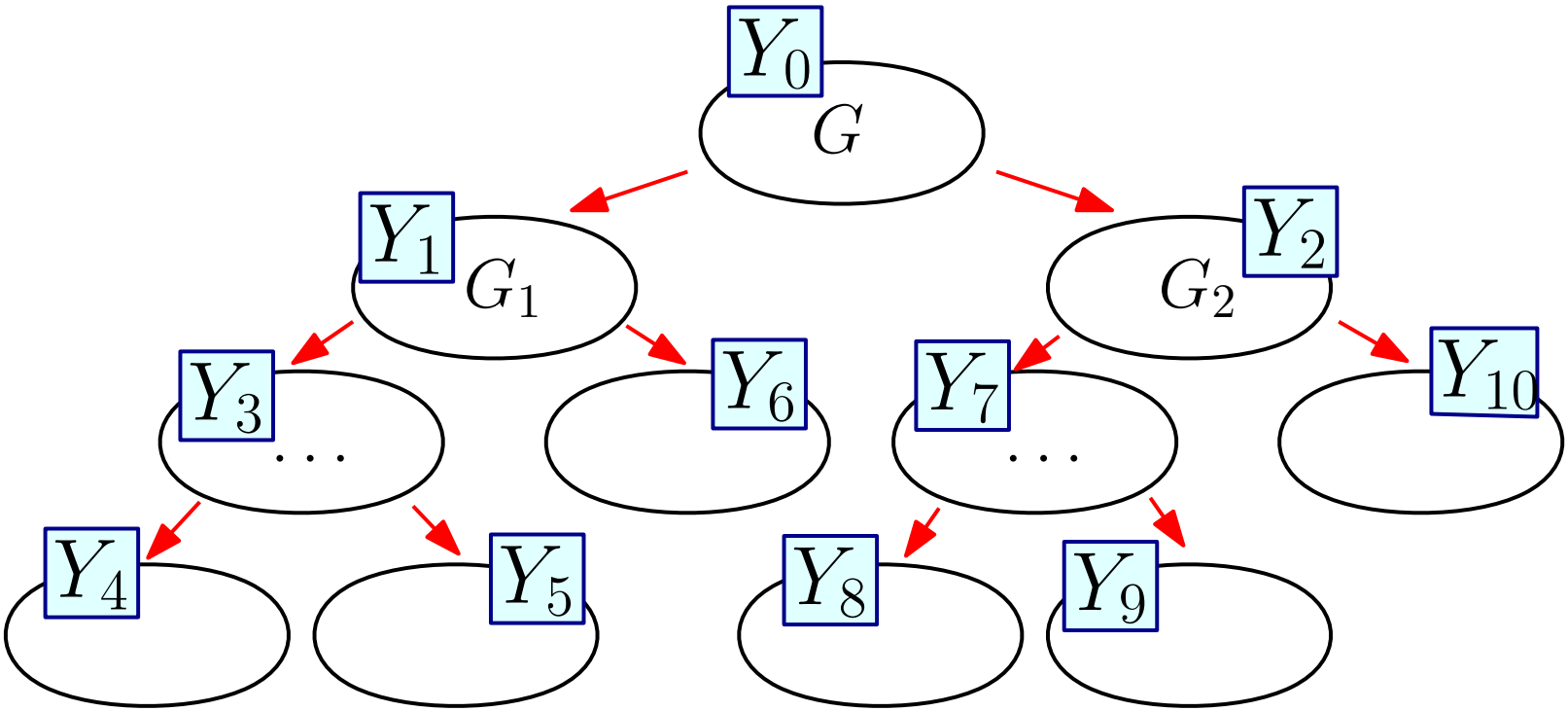
How to bound the size of the tree?

Label each node of the tree with some well-chosen $Y \subseteq V(G)$:



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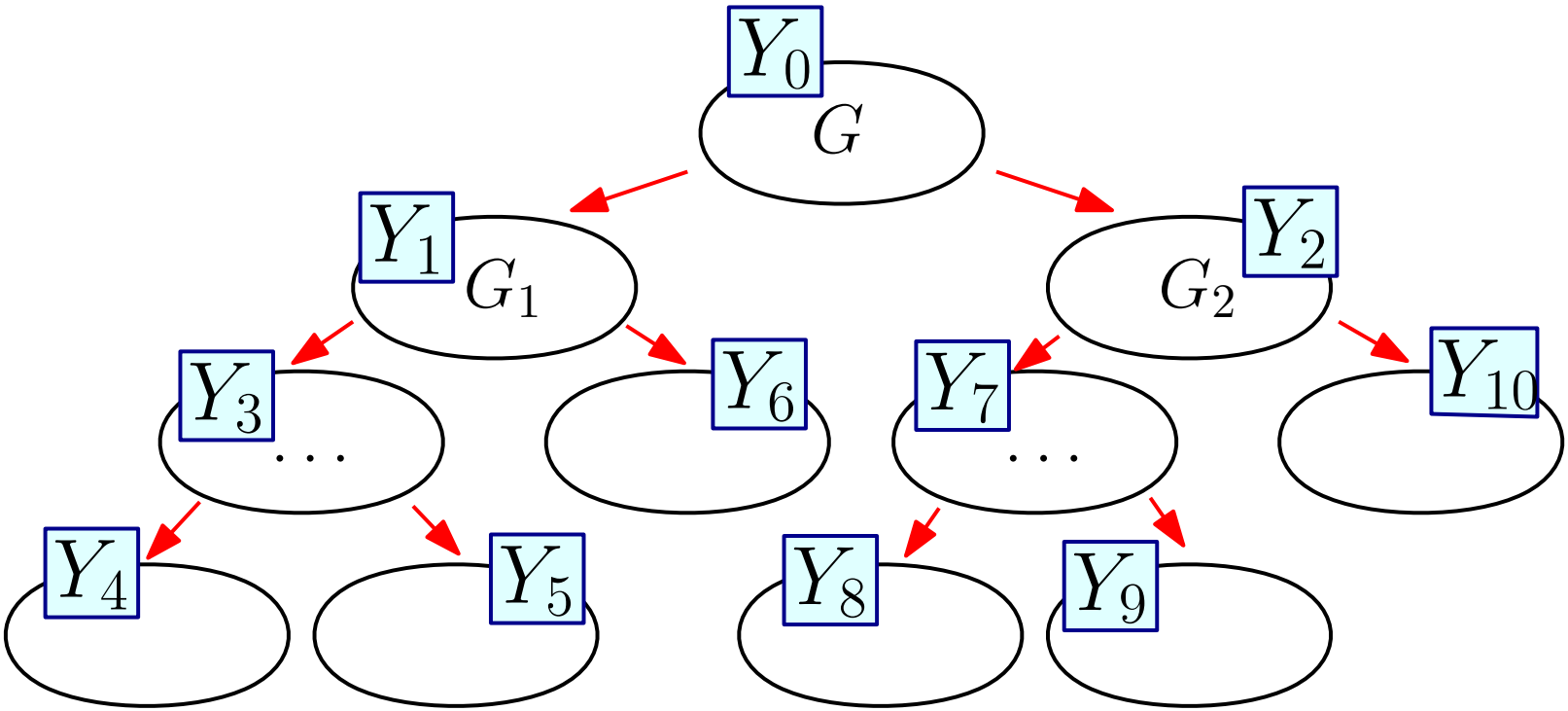

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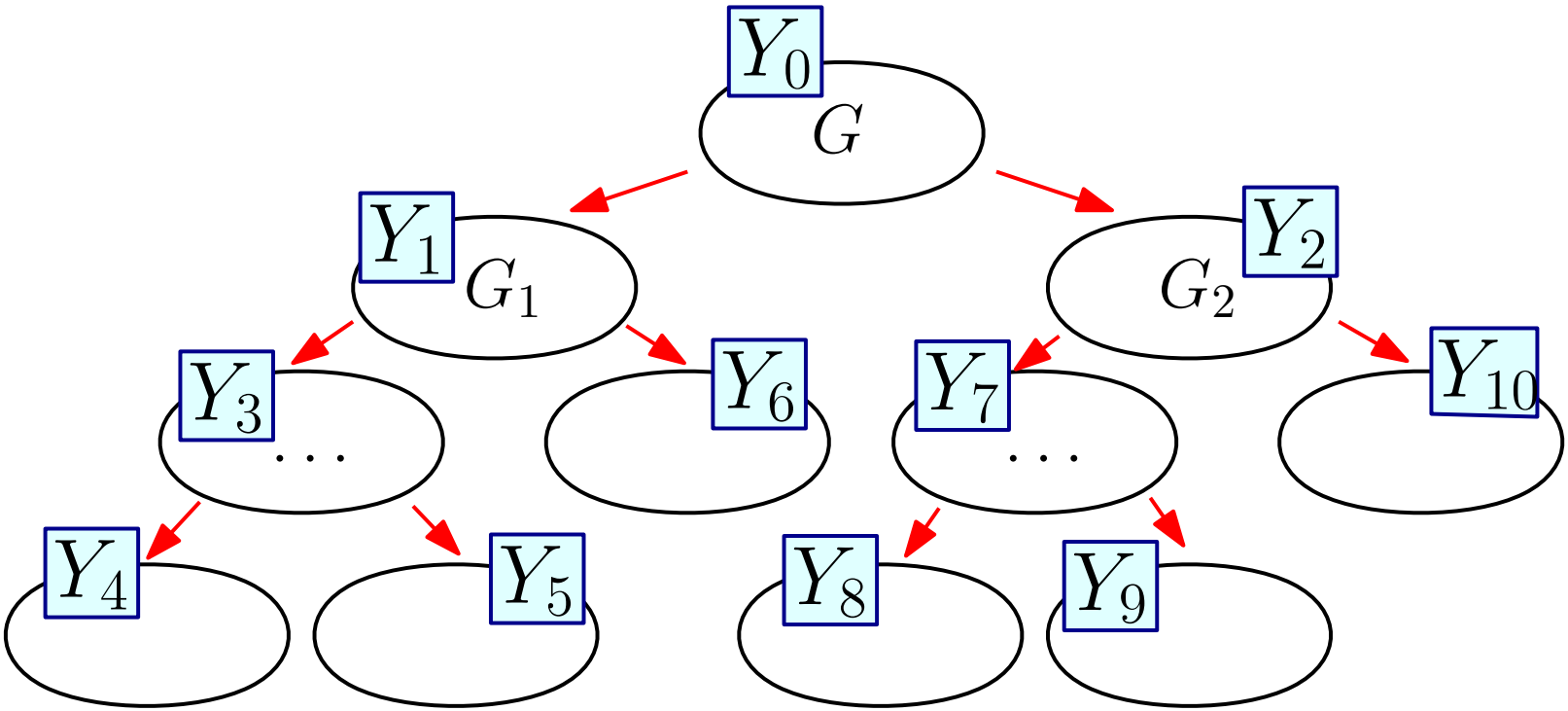

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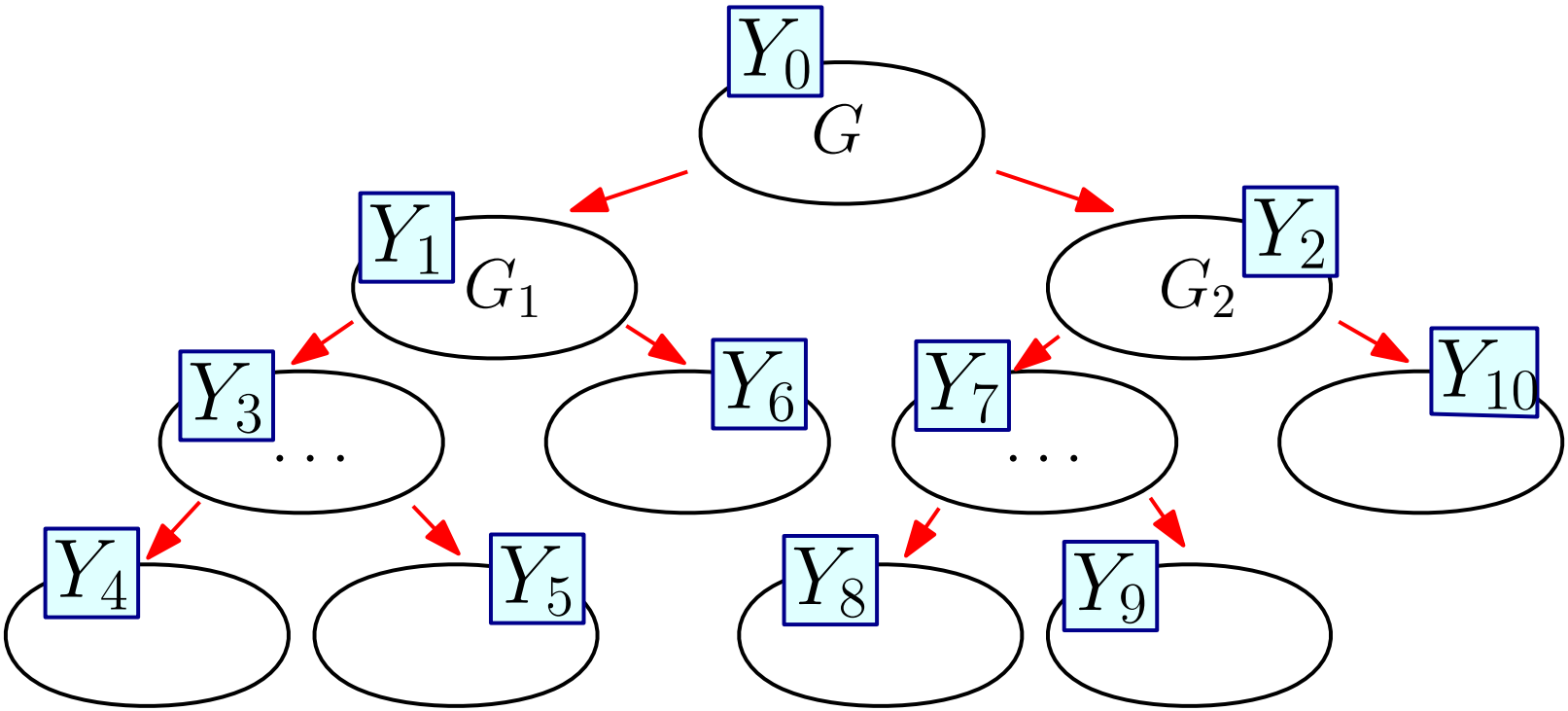

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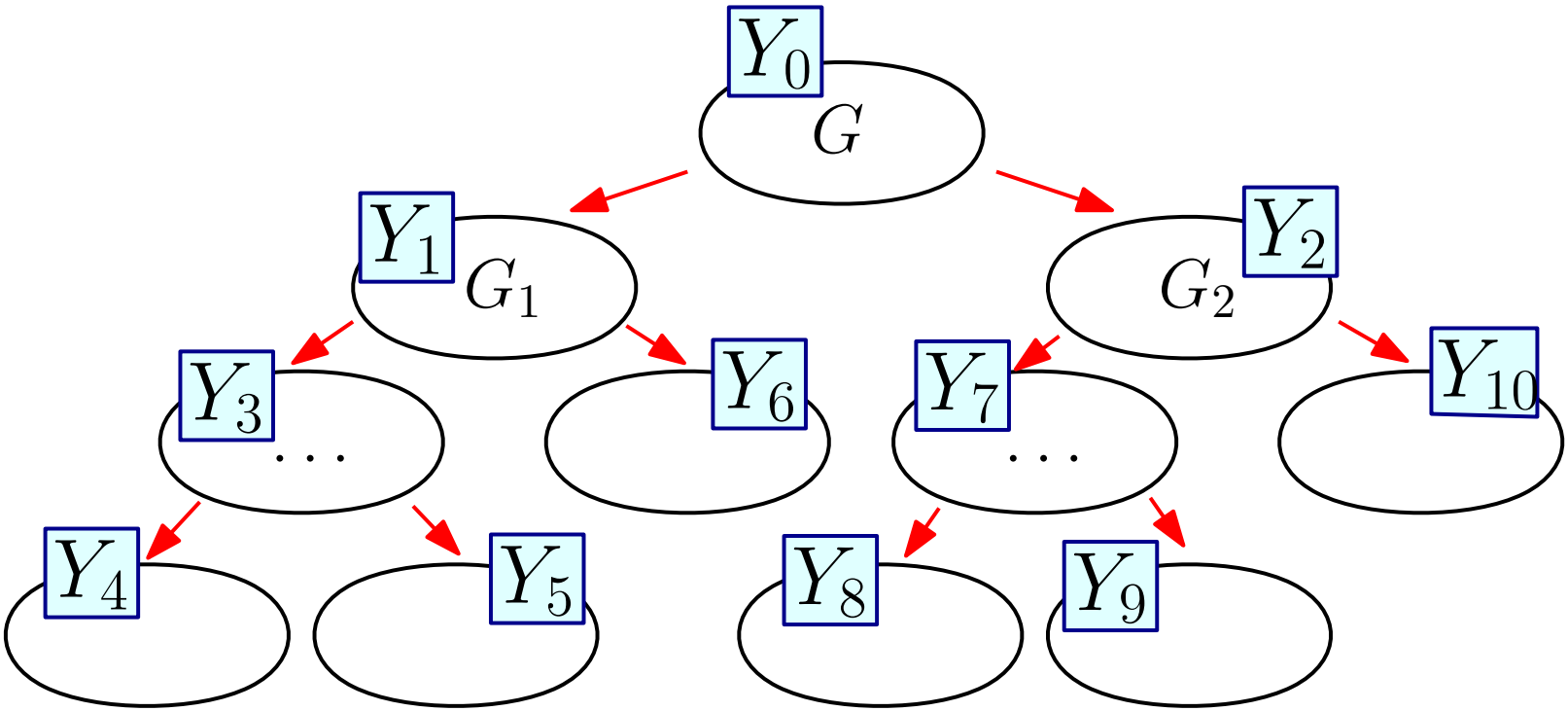

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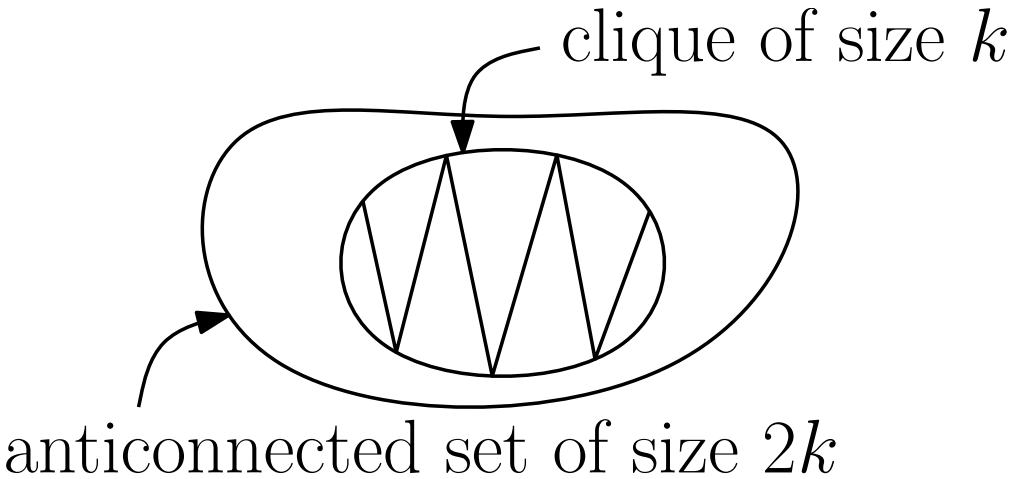
$\Rightarrow$ Bounds the number of nodes by a polynomial.

Key ingredient: k-pellet

Definition: k-pellet

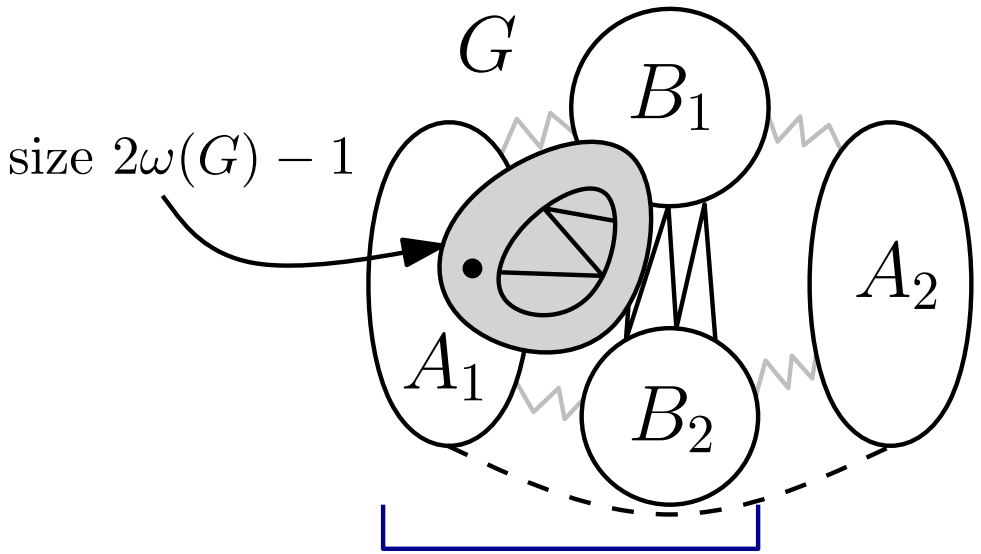
A subset $Y \subseteq V(G)$ is a k-pellet if

- Y contains a clique of size k,
- Y is anticonnected,
- and $|Y| = 2k$.



Number of $\omega(G)$-pellets: at most $\mathcal{O}(n^{2\omega(G)})$!!

At each node decomposing along a BSP, arbitrarily choose an $\omega(G)$-pellet that will be broken.



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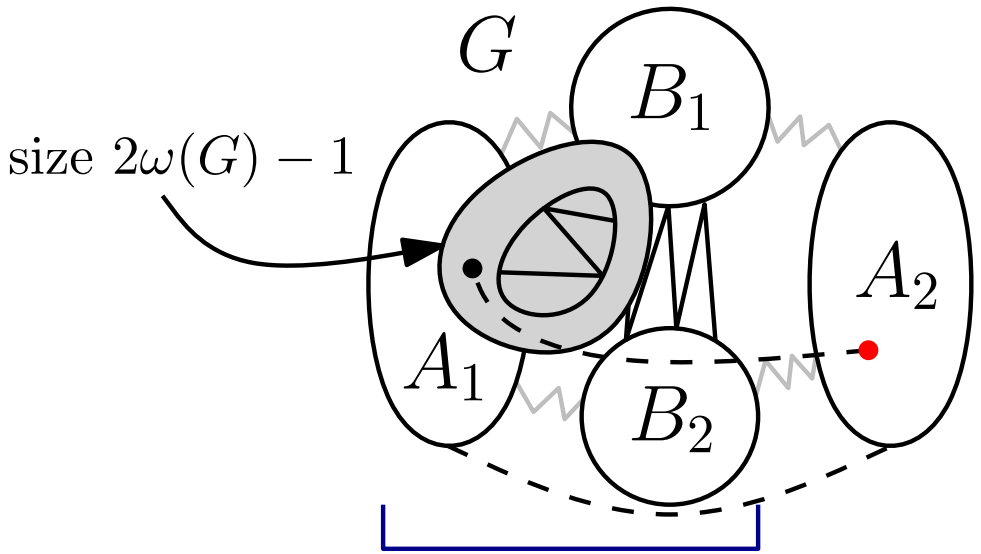

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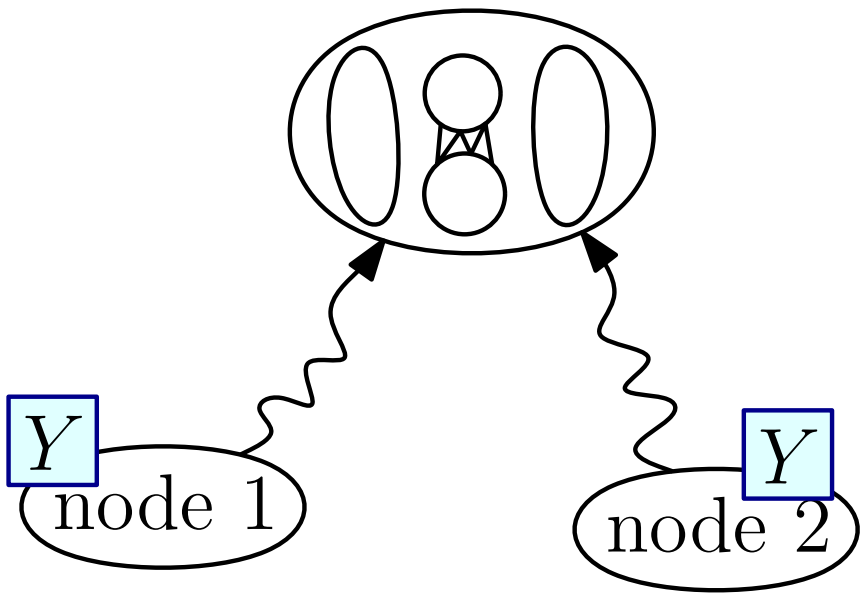

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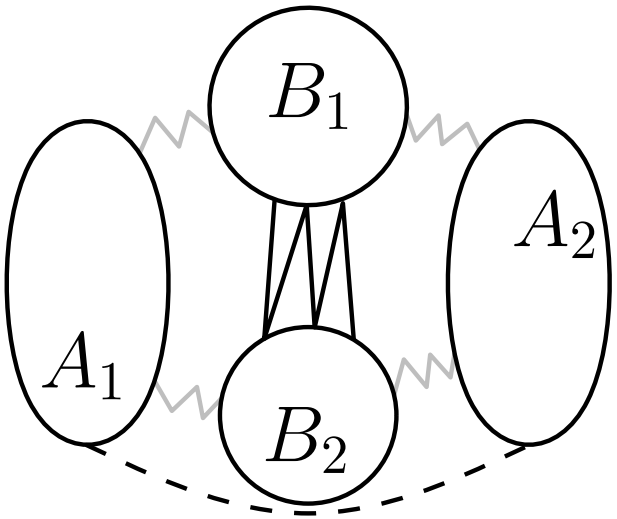

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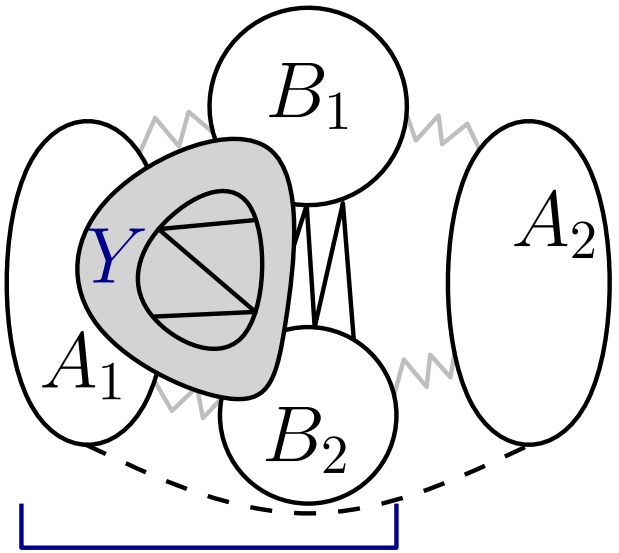

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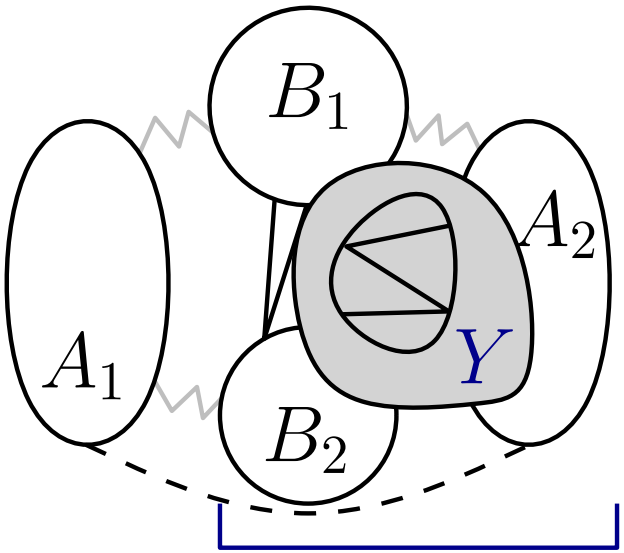

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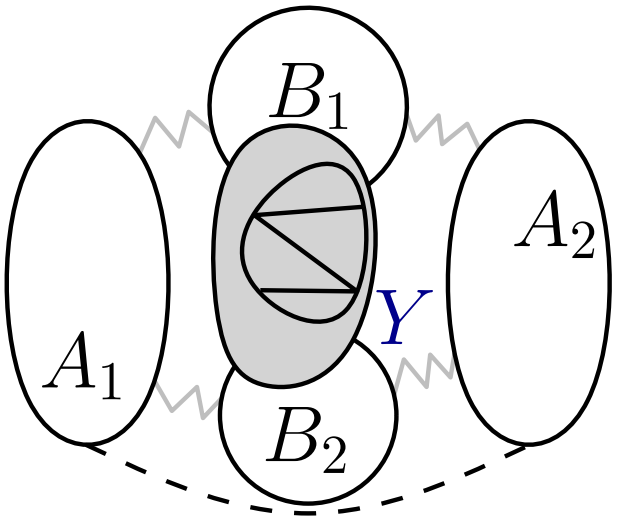

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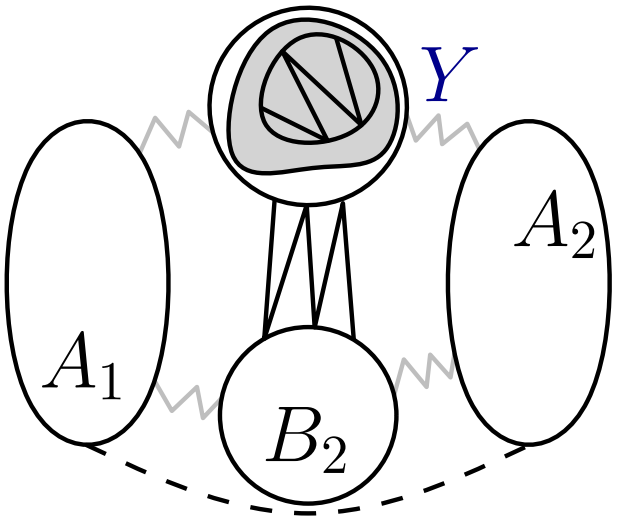

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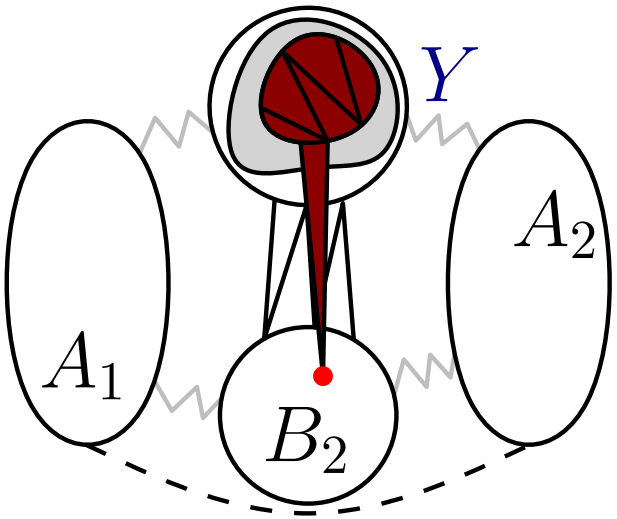

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Y contains a clique of size $\omega(G)$ and any $v \in B_2$ is complete to it.
$\Rightarrow$ Contradiction!

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Previous results:

- Deciding if a graph has a BSP is NP-complete. [Trotignon 08]
- A poly-time algorithm that decides if a **perfect** graph has a BSP can be done in polynomial-time (but, if yes, the algo does not output such a partition). [Trotignon 08]
- A poly-time algo that decides if a graph has a skew partition and, if yes, outputs such a partition. [Kennedy & Reed 08]

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Thank you for your attention!