Polytope Approximation and the Mahler Volume

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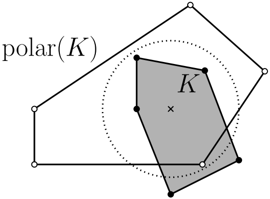
The Mahler Volume



- K: convex body
- Polar body of K: set of points p such that $p \cdot q \leq 1$ for $q \in K$
- Mahler volume of K: product of the volume of K and the volume of $polar(K)$

Important for us:

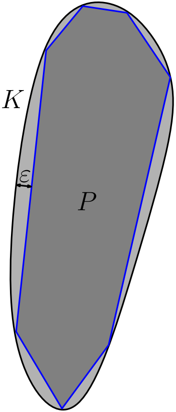
The Mahler volume of K is bounded below by a constant [Kup08]

- Conjecture: A regular simplex attains the minimum volume
- Vast literature for centrally symmetric convex bodies

Polytope Approximation

Problem description:

- Input: convex body K in d-dimensional space and parameter ε
- Output: polytope P which ε-approximates K with a small number of facets (alternatively, vertices)



- Focus on Hausdorff metric in Euclidean spaces of constant dimension d
- Assume (without loss of generality) that $\text{diam}(K) = 1$
- Assume the width of K is at least ε, for otherwise the problem instance should be solved in a lower dimensional space.

Uniform vs. Nonuniform Bounds

- Several algorithms to find the "best" polytope for a given input
- How good is this best polytope?

Nonuniform bounds:

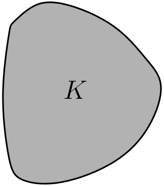
- Hold for $\varepsilon \leq \varepsilon_0$, where ε_0 depends on the input
- Example: Gruber [Gru93] bounds the complexity n using the Gaussian curvature κ of the input

$$n = 1/\varepsilon^{(d-1)/2} \int_{\partial K} \sqrt{\kappa(x)}\, dx$$

Uniform bounds:

- Hold for $\varepsilon \leq \varepsilon_0$, where ε_0 is a constant
- Example: Dudley [Dud74] and Bronshteyn and Ivanov [BI76] bound the maximum number of facets/vertices as a function of ε, d, and the diameter of the input

Dudley's Approximation

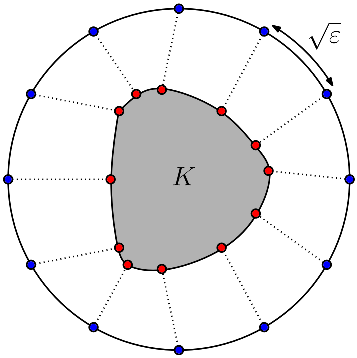


Dudley, 1974:

A convex body K of diameter 1 can be ε-approximated by a polytope P with $O(1/\varepsilon^{(d-1)/2})$ facets.

- Dudley's approximation is the best possible for balls
- It oversamples areas of very high and very low curvatures
- Intuition: A skinny body should be easier to approximate

Dudley's Approximation



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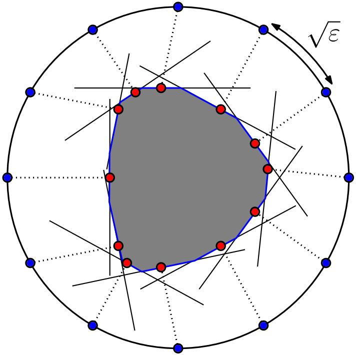

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Our Result: Improved Polytope Approximation

Better uniform bound for *skinny* bodies:

A convex body K can be ε-approximated by a polytope P with $\widetilde{O}(\sqrt{\text{area}(K)}/\varepsilon^{(d-1)/2})$ facets (alternatively, vertices).

- Uses area instead of diameter
- Matches Dudley's bound up to a log factor when the body is fat
- Significant improvement for skinny bodies
- Analysis uses several new techniques for the problem (polarity, Mahler volume, ε-nets...)

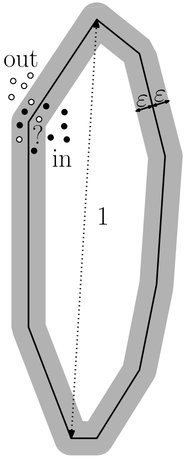
Impact to Other Problems

Approximate polytope membership

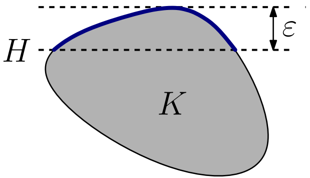
- For the same storage, the query time is reduced to the square root!
- $\widetilde{O}(1/\varepsilon^{(d-1)/8})$ query time with $O(1/\varepsilon^{(d-1)/2})$ (Dudley's) storage

Approximate nearest neighbor (ANN)

- ANN reduces to polytope membership [AFM11]
- Improved query time for storage between $O(n/\varepsilon^{d/4})$ and $O(n/\varepsilon^{d-1})$



Caps and ε-Caps



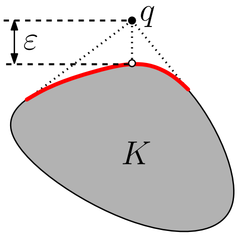
- A cap is the intersection of the boundary of K and a halfspace H.
- The width of a cap C is the maximum distance between a point in C and the boundary of H.
- We refer to a cap of width ε as an ε-cap.

A set N of points stabs all ε-caps if for every cap C we have $C \cap N \neq \emptyset$

Lemma:

If a set N of points stabs all ε-caps, then the convex hull of N is an ε-approximation to K.

Dual Caps and ε-Dual Caps

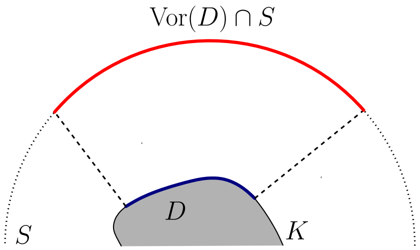


- Dual cap: portion of the boundary of K visible from a point q
- Width: distance between K and q
- ε-dual cap: dual cap of width ε
- A set N of points stabs all ε-dual caps if for every dual cap D we have $D \cap N \neq \emptyset$

Lemma:

If a set N of points stabs all ε-dual caps, then the polytope defined by tangent hyperplanes constructed at the points of N is an ε-approximation to K.

Voronoi Patches



- The Voronoi region Vor(D) of a dual cap D is the set of points closer to D than to any other points of K
- Dudley sphere: Sphere S of radius 3 centered at the origin
- The Voronoi patch of a dual cap D is the intersection Vor(D) $\cap$ S

Mahler Comes In

We show that:

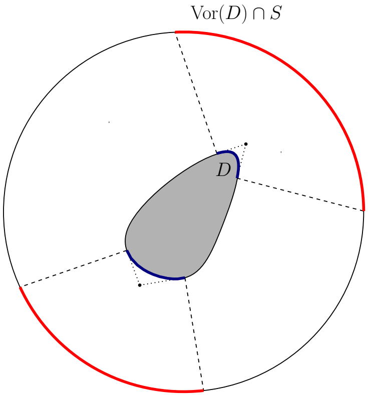
An ε-dual cap D and its Voronoi patch are related in a manner that is similar to the polar transform (up to an ε-scaling).

Using the fact that the Mahler volume is at least a constant:

Key lemma:

For any ε-dual cap D, the product of $\text{area}(D)$ and $\text{area}(\text{Vor}(D) \cap S)$ is $\Omega(\varepsilon^{d-1})$.

Less formally: If D has small area, then its Voronoi patch is large



Stabbing Large ε-Dual Caps

- Large ε-dual cap:

$$\text{area}(D) \geq \sqrt{\text{area}(K)} \cdot \varepsilon^{(d-1)/2}$$

- Fraction of the boundary of K covered by D:

$$\alpha = \frac{\sqrt{\text{area}(K)} \cdot \varepsilon^{(d-1)/2}}{\text{area}(K)} = \frac{\varepsilon^{(d-1)/2}}{\sqrt{\text{area}(K)}}$$

- We stab large ε-dual caps with an α-net on the boundary of the convex body
- Using VC-dimension arguments the size of the α-net is

$$O\left(\frac{1}{\alpha}\log\frac{1}{\alpha}\right) = \widetilde{O}\left(\frac{\sqrt{\text{area}(K)}}{\varepsilon^{(d-1)/2}}\right)$$

Stabbing Small ε-Dual Caps

- Small ε-dual cap: $\text{area}(D) < \sqrt{\text{area}(K)} \cdot \varepsilon^{(d-1)/2}$
- By the key lemma, the Voronoi patch of D is large:

$$\text{area}(\text{Vor}(D) \cap S) > \frac{\varepsilon^{(d-1)/2}}{\sqrt{\text{area}(K)}}$$

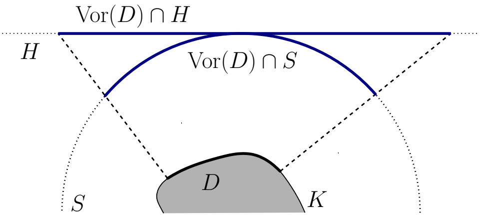
- Dudley ball S has constant area
- Fraction of the boundary of S covered by $\text{Vor}(D) \cap S$:

$$\alpha = \frac{\varepsilon^{(d-1)/2}}{\sqrt{\text{area}(K)}}$$

- We stab small ε-dual caps indirectly, using an α-net on the boundary of the Dudley ball and mapping the points back to K
- Using VC-dimension arguments the size of the α-net is

$$O\left(\frac{1}{\alpha}\log\frac{1}{\alpha}\right) = \widetilde{O}\left(\frac{\sqrt{\text{area}(K)}}{\varepsilon^{(d-1)/2}}\right)$$

Dudley Hyperplane



- In order to bound the area of the Voronoi patch, we first consider a simpler patch
- Dudley hyperplane: Hyperplane H tangent to the Dudley sphere and perpendicular to the direction of the width of the dual cap
- We refer to the intersection $\mathrm{Vor}(D) \cap H$ as the Voronoi patch on H

Approximate Polytope Membership Queries

The same methods improve the existing bounds [AFM11] for:

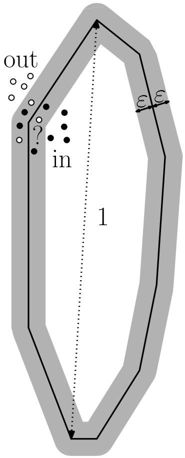
Polytope membership queries

Given a polytope P in d-dimensional space, preprocess P to answer membership queries:

Given a point q, is $q \in P$?

Approximate version

- An approximation parameter ε is given (at preprocessing time)
- Assume the polytope has diameter 1
- If the query point's distance from P's boundary:
  - $> \varepsilon$: answer must be correct
  - $\leq \varepsilon$: either answer is acceptable



Approximate Polytope Membership Queries

Improved tradeoff for approximate polytope membership queries

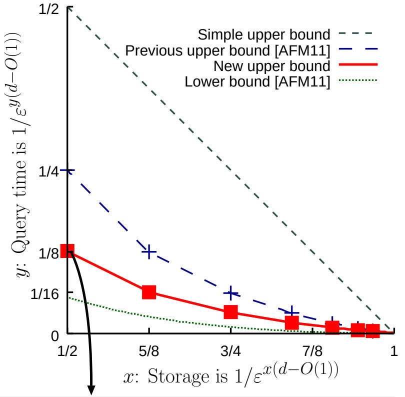
New bounds:

For integer $k \geq 2$, we can answer ε-approximate polytope membership queries with

Storage: $O(1/\varepsilon^{(d-1)/(1-k/2^k)})$

Query time: $O(1/\varepsilon^{(d-1)/2^{k+1}} \log(1/\varepsilon))$

- For the same storage, the query time is reduced to roughly the square root
- Leads to improved approximate nearest neighbor data structures



Storage: $O(1/\varepsilon^{(d-1)/2})$

Query time: $\widetilde{O}(1/\varepsilon^{(d-1)/8})$

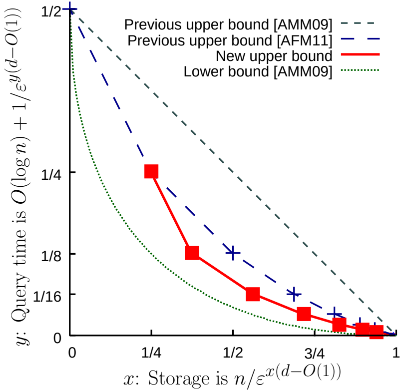
Restricted Dual Caps

- To prove improved bounds for approximate polytope membership queries we need to generalize the key lemma
- The proof follows the same outline (but gets much more intricate)
- A restricted ε-dual cap is the intersection of an ε-dual cap and a ball of radius $\sqrt{\varepsilon}$

Stronger key lemma:

Let D' be a restricted ε-dual cap. The product of $\text{area}(D')$ and $\text{area}(\text{Vor}(D') \cap S)$ is $\Omega(\varepsilon^{d-1})$.

Approximate Nearest Neighbor (ANN) Searching



- ANN: Preprocess n points such that, given a query point q, can find a point within at most $1+\varepsilon$ times the distance to q's nearest neighbor
- It is possible to reduce ANN to approximate polytope membership [AFM11]
- Improved query time for storage between $O(n/\varepsilon^{d/4})$ and $O(n/\varepsilon^{d-1})$

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Thank you!